*Proc. of the 9th International Symposium on Geotechnical Safety and Risk (ISGSR)*
*Edited by* Zhongqiang Liu, Jian Dai and Kate Robinson
*Published by* Research Publishing, Singapore. 
doi: 10.3850/GRF-25280825_isgsr-213-P149-cd

ISGSR2025

# SHAPE FUNCTION-BASED KL EXPANSION METHOD FOR DISCRETIZINGIRREGULAR RANDOM FIELDS

Zhihao JIANG
School of Resources and Environmental Engineering, Hefei University of Technology, Hefei 230009, China
E-mail: jiangzh@mail.hfut.edu.cn

Xiaohui TAN
School of Resources and Environmental Engineering, Hefei University of Technology, Hefei 230009, China
E-mail: tanxh@hfut.edu.cn

Shanwei LIU
School of Resources and Environmental Engineering, Hefei University of Technology, Hefei 230009, China
E-mail: shanweiliu@whu.edu.cn

Xiaoliang HOU
School of Resources and Environmental Engineering, Hefei University of Technology, Hefei 230009, China
E-mail: xlhou@hfut.edu.cn

Abstract*:*The Karhunen-Loève (KL) expansion is widely used to simulate soil spatial variability but faces challenges in efficiently discretizing irregular random fields.A shape function-based KL (SFKL) discretization method is proposed in this study, where shape functions are combined with Lagrange interpolation to transform multidimensional integrals into matrix assembly operations. This approach significantly improves the computational efficiency and accuracy while addressing the limitations of OSEmethods forirregulardomains. Case studies demonstrate the effectiveness of the proposed method, showing that the SFKL method achieves lower discretization errors and requires fewer expansion terms than the OSE method in irregular domains.

*Keywords*: Random field, Karhunen-Loève expansion, shape functions, slope, spatial variability, irregular domain.

## 1. Introduction

The Karhunen-Loève (KL) expansion has been extensively applied in geotechnical reliability analysis for modeling soil spatial variability(Phoon et al., 2002;Zhu et al., 2021). The method involves solving the integral eigenvalue problem (IEVP) of the autocorrelation function (ACF) through numerical discretization, typically implemented via the Galerkin approach(Liu and Zhang, 2017).In the Galerkin method, basis functions such as Legendre polynomials are chosen to span the entire integration domain, allowing the corresponding integral equations to be solved efficiently.However, when applied to random field discretization, irregular domains are often approximated as regular regions, leading to unnecessary errors and an increased number of expansion terms(Allaix and Carbone, 2013).

This study integrates shape functionswith Lagrange interpolation to simplify multidimensional integral to the assembly of simple matrices, enhancing the computational efficiency and accuracy of KL expansion method for irregular random field discretization. The method is validated through twocase studies of both regular and irregular random domains.

## 2. KL Expansion Method

### 2.1.*Basic concept for KL expansion method*

The KL expansion discretely represents the spectral decomposition of theACF $\rho(\boldsymbol{x},\boldsymbol{x}')$, allowing a random field to be expressed as shown in Eq. (1)(Betz et al, 2014):

$$H(\boldsymbol{x},\theta)=\mu(\boldsymbol{x})+\sigma(\boldsymbol{x})\sum_{i=1}^{\infty}\sqrt{\lambda_i}\varphi_i(\boldsymbol{x})\xi_i(\theta) \tag{1}$$

where $\mu(\boldsymbol{x})$ and$\sigma(\boldsymbol{x})$are the mean and standard deviation function;$\xi_i(\theta)$ are independent random variables; $\lambda_i$ and $\varphi_i(x)$ are the eigenvalues and eigenfunctions of the ACF, which are obtained by solving the IEVP as shown in Eq. (2)(Betz et al, 2014):

$$\int_D \rho(\boldsymbol{x},\boldsymbol{x}')\varphi_i(\boldsymbol{x})\mathrm{d}\boldsymbol{x}=\lambda_i\varphi_i(\boldsymbol{x}') \tag{2}$$

where $\boldsymbol{x}$ and $\boldsymbol{x}'$ are the coordinates of spatialpoints. Commonly used ACFs in geotechnical engineering are the exponential, Gaussian, second-order autoregressive, and exponential cosine functions.

#### 2.1.1.*Analytical solution for the KL method (KLA)*

IEVP can be solved analytically under certain specific conditions.Ghanem and Spanos (1991)provided solutions for the exponential ACF(see Eq. (3)), where $L_h$ is the autocorrelation distance (ACD) of the random field.

$$\rho\left(\boldsymbol{x},\boldsymbol{x}'\right)=\exp\left(-\left|\boldsymbol{x}-\boldsymbol{x}'\right|/L_{\mathrm{h}}\right) \tag{3}$$

#### 2.1.2.*Orthogonal series expansion for the KL method (OSE)*

In the Galerkinmethod, the eigenfunctions$\varphi_i(x)$ is expressed as a linear combination of a series of orthogonal basis functions $h_k(x)$ ($k$=1, 2, …, $N$) as follows (Lin et al, 2022):

$$\varphi_i\left(\boldsymbol{x}\right)\approx\hat{\varphi}_i\left(\boldsymbol{x}\right)=\sum\nolimits_{k=1}^{N}D_k^{(i)}h_k(\boldsymbol{x})=[\boldsymbol{D}^{(i)}]^{\mathrm{T}}\boldsymbol{h}(\boldsymbol{x}) \tag{4}$$

where $\boldsymbol{D}^{(i)}$ is a column vector of $N$×1 whose elements are the undetermined coefficient $D_k^{(i)}$ ($N$ is the number of orthogonal basis function terms; and the superscript 'T' represents the transpose of a matrix). Substitute Eq.(4) into Eq.(2)and apply orthogonalization with the $h_k(x)$to express the result in matrix form, as shown in Eq. (5)(Lin et al, 2022):

$$\boldsymbol{AD}=\boldsymbol{ED\Lambda}\Rightarrow\boldsymbol{E}^{-1}\boldsymbol{AD}=\boldsymbol{D\Lambda} \tag{5}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix of the approximate eigenvalues; $\boldsymbol{A}$ and $\boldsymbol{E}$ are both symmetric matrices of $N$×$N$ whose elements are shown in the following equations(Lin et al, 2022):

$$\boldsymbol{A}_{kl}=\int_{\Omega}\int_{\Omega}\rho\left(\boldsymbol{x},\boldsymbol{x}'\right)h_k\left(\boldsymbol{x}\right)h_l\left(\boldsymbol{x}'\right)\mathrm{d}x\mathrm{d}x',\boldsymbol{E}_{kl}=\int_{\Omega}h_k\left(\boldsymbol{x}\right)h_l\left(\boldsymbol{x}'\right)\mathrm{d}x \tag{6}$$

It is worth noting that to ensure a successful solution, the basic condition $M \le N$ should be met.

### 2.2.*Shape function-based KL Expansion Method*(SFKL)

To simplify the integration of the ACF, the discretized domain is initially discretized into finite elements (subregions $\Omega_e$). For each element, the global coordinates $x$is converted to local coordinates $\xi$defined on the reference domain [-1,1] through a transformation mapping.Shape functions are piecewise polynomial approximations defined on local coordinates which are used to describe the variation of variables for each element(Pozrikidis, 2005). For multidimensional elements (e.g., 2D quadrilaterals and 3D hexahedra),the shape functionsare constructed by combining 1D shape functions for each dimension.Taking 2D quadrilateral elements as an example,the construction of the shape functions can be expressed as follow:

$$\boldsymbol{N}\left(\xi,\eta\right)=\boldsymbol{N}\left(\xi\right)\otimes\boldsymbol{N}\left(\eta\right)=\begin{bmatrix}(1-\xi)/2\\(1+\xi)/2\end{bmatrix}\otimes\begin{bmatrix}(1-\eta)/2\\(1+\eta)/2\end{bmatrix}=\begin{bmatrix}(1-\xi)(1-\eta)/4 & (1-\xi)(1+\eta)/4\\(1+\xi)(1-\eta)/4 & (1+\xi)(1+\eta)/4\end{bmatrix} \tag{7}$$

Due to the complexity of the ACF, it is approximated using an m-dimensional Lagrange interpolation polynomial (Lin et al., 2022). By replacing the original basis functions with shape functions, the Eq.(6) can be written as:

$$\boldsymbol{A}\approx\boldsymbol{K}^{\mathrm{T}}\boldsymbol{\rho}\boldsymbol{K}=\left(\sum\nolimits_{e=1}^{n}\int_{\Omega_e}\boldsymbol{L}(\boldsymbol{x})\boldsymbol{N}(\boldsymbol{x})\mathrm{d}x\right)^{\mathrm{T}}\boldsymbol{\rho}\left(\sum\nolimits_{e=1}^{n}\int_{\Omega_e}\boldsymbol{L}(\boldsymbol{x})\boldsymbol{N}(\boldsymbol{x})\mathrm{d}x\right),\boldsymbol{E}\approx\sum\nolimits_{e=1}^{n}\int_{\Omega_e}\boldsymbol{N}(\boldsymbol{x})\boldsymbol{N}(\boldsymbol{x}')\mathrm{d}x \tag{8}$$

where $\boldsymbol{L}(\boldsymbol{x})$ is the element Lagrange interpolation function (which satisfies $L_i(x_j)=\delta_{ij}$);$\boldsymbol{N}(\boldsymbol{x})$ is the element shape functionsmatrix;$n$ is the number of discretized subregions $\Omega_e$; $\boldsymbol{\rho}$ isan autocorrelation matrix of Lagrange interpolation points. The integral can then be converted into matrix computations using the Gaussian quadrature method.

Discretization error in the KL expansion, due to truncation and eigenvalue approximation, can bequantified using the absolute covariance error. Formally, it is defined asshown in Eq.(9),where $V$ is the random field domain:

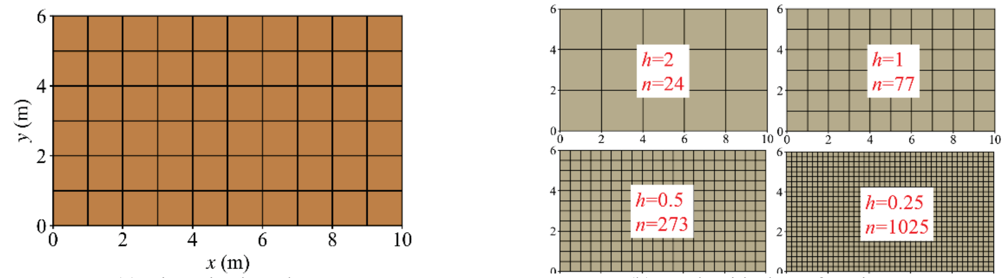
$$\overline{\varepsilon}_{\mathrm{cov},M}\left(\boldsymbol{x}\right)=\frac{1}{V^2}\iint_{\Omega\,\Omega}\left|\rho(\boldsymbol{x},\boldsymbol{x}')-\sum_{i=1}^{M}\hat{\lambda}_i\hat{\varphi}_i(\boldsymbol{x})\hat{\varphi}_i(\boldsymbol{x}')\right|\mathrm{d}x\mathrm{d}x' \tag{9}$$

## 3. Case Studies

The two 2D case studies demonstrate the advantages and versatility of the proposed method, which can also be extended to 3D irregular random field problems.

### 3.1. *Regular rectangular region*

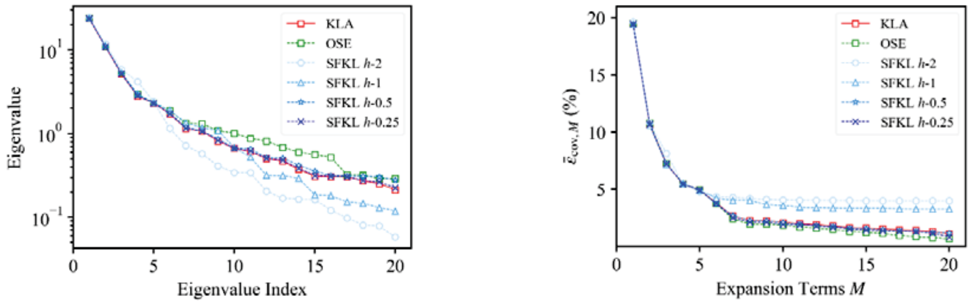
The case study examines a 2D rectangular domain typical of shallow foundation analysis.The soil cohesion, following a log-normal distribution, has a mean value of 34 kPa and a coefficient of variation (CV) of 0.3, with a single exponential ACF.The domain measures 10m by 6m, and the soil's autocorrelation distances are 20m horizontally and 2m vertically. The rectangular domain is discretized into 60 grids whose size is 1•×1m, as shown in Fig. 1(a). In Fig. 1(b), *h* denotes the length of girdand *n* denotes the number of shape functions in the mesh.




(a) Discretized mesh (b) Mesh with shape functions

Fig. 1. Schematic diagram of regular rectangular region

Fig. 2(a)compares the eigenvalues computed using different KL methods, showing that the SFKL method closely approximates the analytical solution as the number of shape functions increases.Fig. 2(b)further illustrates that increasing the number of shape functions in the mesh helps reduce discretization error.

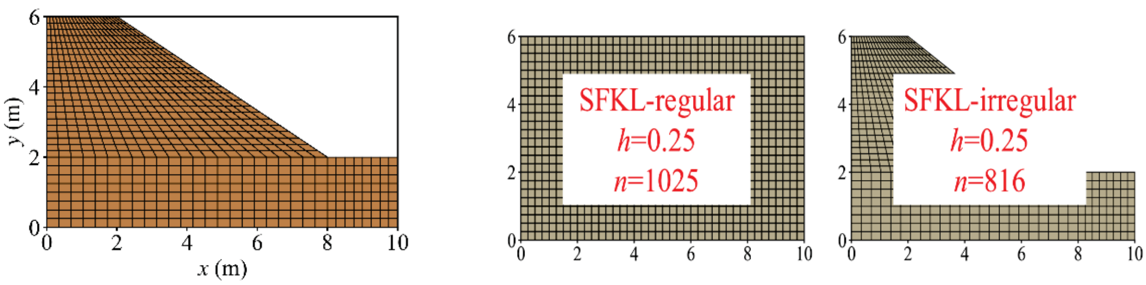



(a) Eigenvalue results (b) Error results

Fig. 2. Comparison between eigenvalue and error of the three methods

### 3.2. *Irregular region*

To demonstrate the applicability of the SFKL to irregular domains, a 2D slope model is studied. This model is obtained by removing a portion of the regular domain from Case 3.1.To compare the SFKL with the OSE method, the mesh with shape functions is generated based on both the original irregular domain mesh and irregular domain mesh, as shown inFig. 3(b).

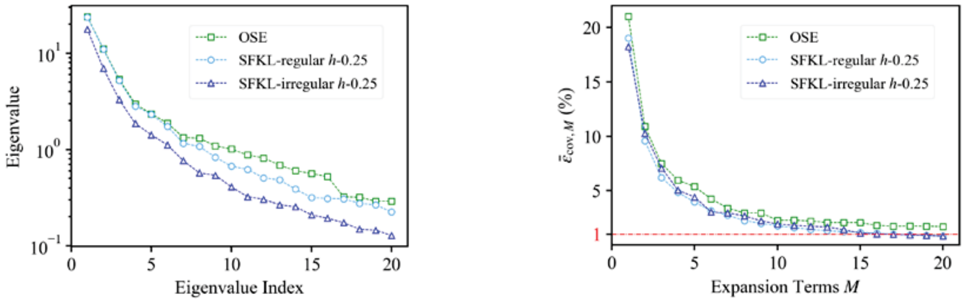



(a) Discretized mesh (b) Mesh with shape functions

Fig. 3. Schematic diagram of irregular region

Fig. 4(a)compares the computational results obtained by the SFKL method and the OSE method. The eigenvalues calculated by the OSE method show good agreement with those obtained by the SFKL method in the regular domain, but exhibit overestimation when compared to the SFKL results in the irregular domain This suggests that the OSE method does not fully account for the impact of the irregular domain on the eigenvalues.

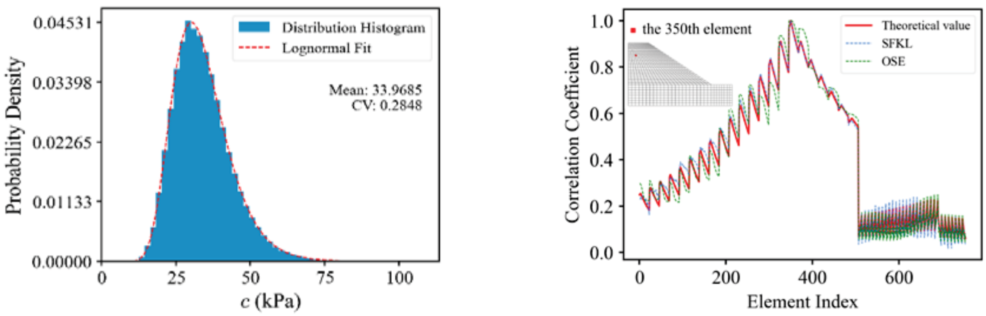
As shown in Fig. 4(b), for a 1% error threshold, the SFKL method attains the desired accuracy with only 16 expansion terms, whereas the OSE method fails to meet this threshold within the first 20 terms. This underscores the faster convergence of the SFKL method, which achieves the same level of accuracy with fewer expansion terms. Consequently, the SFKL method exhibits superior efficiency and precision, particularly when dealing with irregular domains.




(a) Eigenvalue results (b) Error results

Fig. 4. Comparison between eigenvalue and error of irregular region

Fig. 5(a) illustrates the statistical distribution of the soil cohesion values after conducting 1000 random samplings, which have been discretized utilizing the SFKL method. It is evident that both the mean and CV closely align with the predefined values. Furthermore,Fig. 5(b) displays the variation in the correlation coefficient of the 350th element in the slope model compared to the other elements. Notably, the SFKL method offers a more accurate approximation to the theoretical values than the OSE method. These findings demonstrate that the SFKL method effectively realizes the spatial variability of soil parameters.




(a) Statistical distribution (b) Correlation coefficient variation

Fig. 5. Parameter Distribution and Correlation in the Slope Model

## 4. Conclusions

The spatial variability in both regular and irregular domains can be effectively simulated by the SFKL method, simplifying ACF computations without sacrificing accuracy. The SFKL method closely aligns with analytical solutions in regular domains, and it outperforms the OSE method and achieves considerable accuracy with fewer expansion terms in irregular domains. Generally, the SFKL method provides an efficient and accurate solution for representing random fields, especially in the existence of irregular boundaries. Although the SFKL method is highly applicable, future work will focus on optimizing it to improve computational efficiency in large-scale 3D simulations for more complex engineering problems.

**Acknowledgement**

This work wassupportedbytheNationalNaturalScienceFoundationofChina(GrantNo. 42372302).

**References**

Phoon, K. K., Huang, S. P., and Quek, S. T. (2002). Implementation of Karhunen–Loeve expansion for simulation using a wavelet-Galerkin scheme. *Probabilistic Engineering Mechanics*, 17(3), 293–303.

Zhu, B., Hiraishi, T., Pei, H., and Yang, Q. (2021). Efficient reliability analysis of slopes integrating the random field method and a Gaussian process regression-based surrogate model. *International Journal for Numerical and Analytical Methods in Geomechanics*, 45(4), 478–501.

Liu, Q., and Zhang, X. (2017). A Chebyshev polynomial-based Galerkin method for the discretization of spatially varying random properties. *Acta Mechanica*, 228(6), 2063–2081.

Allaix, D. L., and Carbone, V. I. (2013). Karhunen–Loève decomposition of random fields based on a hierarchical matrix approach. *International Journal for Numerical Methods in Engineering*, 94(11), 1015–1036.

Ghanem, R. G., and Spanos, P. D. (1991). *Stochastic Finite Elements:A Spectral Approach*. Springer.

Lin, X., Tan, X., Yao, Y., Dong, X., Fei, S.,and Ma, L. (2022). Realization of multi-dimensional random field based on Jacobi–Lagrange–Galerkin method in geotechnical engineering. *Computers and Geotechnics*, 144, 104533.

Betz, W., Papaioannou, I., and Straub, D. (2014). Numerical methods for the discretization of random fields by means of the Karhunen–Loève expansion. *Computational Methods in Applied Mechanics and Engineering*, 271, 109–129.

Pozrikidis, C. (2005) *Finite and spectral element methods using Matlab*. CRC Press.