Proc. of the 9th International Symposium on Geotechnical Safety and Risk (ISGSR)
Edited by Zhongqiang Liu, Jian Dai and Kate Robinson
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doi: 10.3850/GRF-25280825_isgsr-186-P137-cd

PROBABILISTIC INVERSION OF ELECTRICAL RESISTIVITY TOMOGRAPHY DATA USING HAMILTONIAN MONTE CARLO (HMC) ALGORITHM

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Electrical resistivity tomography (ERT) is being increasingly used in geotechnical site characterisation, dam monitoring, contamination flow study and mining application. To interpret the ERT survey measurements, inversion of measured data is required to be carried out. Electrical resistivity tomography is a non-linear and ill-posed geophysical inverse problem. ERT inversion is ill-posed since the measured data being scarce and inclusive of noise, allows multiple solution sets to fit the data within the uncertainty level of the observation. A probabilistic approach in the Bayesian framework to carry out the inversion of 2D electrical resistivity tomography (ERT) is presented, in which a gradient-based Markov chain Monte Carlo method called the Hamiltonian Monte Carlo (HMC), is used. In this approach, the posterior probability density is evaluated to quantify the uncertainty in the solution. Conventional Markov chain Monte Carlo methods like Metropolis Hastings technique require large number of sampling to obtain a fair estimation of the posterior in higher dimensional parameter spaces. Use of HMC enables efficient statistical estimation over high-dimensional parameter space. HMC uses derivatives of the forward equation, which enables to propose new models far away from the current one, thus generating independent realizations of the posterior while maintaining high acceptance rate. Moreover, the truncated Karhunen-Loève (KL) expansion is employed to enable dimensionality reduction of the spatial field parameters associated with the subsurface electrical conductivity distribution. A 2.5D electrical resistivity forward model was developed for this purpose and used for the inversion. A synthetic problem is solved using the proposed approach and its efficacy is assessed.

Keywords: Electrical Resistivity Tomography, Inversion, Bayesian method, Markov chain Monte Carlo, Hamiltonian Monte Carlo, Karhunen-Loève expansion, Uncertainty quantification.

1. Introduction

Electrical resistivity tomography (ERT), which is frequently used in site characterisation, subsurface monitoring and mining operations, requires inversion of measured field data to obtain the subsurface parameters. Typically, deterministic gradient-based algorithms are employed to invert ERT measurement data. However, deterministic inversion does not capture the uncertainty in the solution. In recent years, probabilistic inversion methods are being used since these methods have the capability to generate diverse geological models, while maintaining a good match with the observed data. This ensures a proper capture of uncertainty in the predictions. The Bayesian setting is usually adopted to accurately propagate the uncertainties from the available geophysical data onto the estimated parameters. Different sampling methods like Metropolis-Hastings Markov chain Monte Carlo (MH-MCMC), Gibbs and Slice sampling have been used in geophysical inversion problems. Hamiltonian Monte Carlo (HMC) has been recently used in some geophysical inversion problems involving seismic, gravity, magnetic, electromagnetic and acoustic signals (Koch et al. 2021, Zunino et al. 2023). In this study, HMC is used for inversion of a synthetic electrical resistivity data and the estimated solution is evaluated.

2. Forward Modeling

A MATLAB-based 2.5-D forward modeling code has been developed in this work, based on the concept proposed by Dey & Morrison (1979). The 3-D potential field due to a known input current is related to the electrical conductivity (EC) structure, for appropriate boundary conditions, as described in Eq. (1) below:

$$\nabla \cdot \left(\boldsymbol{\sigma}(x,y,z)\nabla \mathbf{u}(x,y,z)\right) = -\mathbf{I}\left(\delta(\mathbf{r}-\mathbf{r_s})\right) \text{ in } \Omega \subset \mathbb{R} \qquad (1)$$

where $\boldsymbol{\sigma}(x,y,z)$ represents a given EC distribution of the ground, $\mathbf{I}$ the source current from a dipole and $\mathbf{u}(x,y,z)$ the sought electrical potential. Here, $\mathbf{r_s}$ is the location of the current source and the Dirac's delta δ function forces a singular current density at the source position. With the assumption that the geology is 2-D, Dey & Morrison (1979)

used a Fourier cosine transform to yield an algorithm that considers current flow in all three dimensions, without having to explicitly model in 3-D. This results in an algorithm that captures the 3-D physics but is substantially more efficient to compute than the full 3-D solution. Many researchers have developed forward model based on this concept. Since the current electrode is a point source, total potential is singular at the source point. As a result, results obtained using numerical approximations are poor near the source current location (Pan & Tang, 2014). To remove the effect of the singular potential, an approach introduced by Lowry et al. (1989) is used. According to this singularity removal technique, the total potential is split into primary and secondary potential. The primary potential is that resulting from the source current in a uniform half-space with a constant EC and the secondary potential is the one resulting from the inhomogeneity (i.e. difference between true and constant EC).

Since Fourier transformation is used in 2.5D forward model, the accuracy of the result depends on the number of Fourier coefficients used. Higher the number of Fourier coefficients used, higher is the accuracy; however, it is achieved at the stake of computational effort. Hence, limited number of Fourier coefficients are used which have highest contribution. Pan & Tang (2014) have used a newly developed differential evolution algorithm to solve the non-linear optimization problem to find the optimum Fourier coefficients. The Fourier coefficients thus evaluated are used in this study. Finite element method is employed in the developed forward model, utilizing an unstructured mesh composed of quadrilateral elements and a graded element size in the domain geometry. The potential calculated from the developed forward model are verified with problems having available analytical solution. Based on the attempted verifications, the errors are found to be less than 2%.

3. Inversion

The Bayesian approach, based on Bayes' rule, treats the parameter θ as a random variable. The parameter θ, which is to be estimated, is inferred from the posterior probability distribution. This approach incorporates prior knowledge of the problem through the prior probability density $p(\theta)$, which, when combined with the likelihood, determines the posterior density. The soil parameter being an unknown random variable at each element of the discretized space, the number of unknown variables in geophysical and geotechnical problems are usually very large. Due to the large number of unknowns, solving the inverse problem becomes computationally challenging. In order to reduce the computational effort, dimensionality reduction techniques like Karhunen-Loeve (KL) expansion can be used (Betz et. al. 2014, Koch et al. 2021).

For inverse problems, a numerical evaluation of the posterior can be derived using, for example, Markov Chain Monte Carlo (MCMC) sampling algorithms or ensemble-based methods. Hamiltonian Monte Carlo (HMC) (Neal 2011) is an MCMC method that exploits the concept of conservation of Hamiltonian and enables transitions to distant points in the typical set. The reason for establishing the Hamiltonian is that it enables the generation of trajectories of θ, through Hamiltonian dynamics. This is done by augmenting the posterior density with a momentum variable, p. To numerically integrate the Hamiltonian trajectories, the leapfrog method, which is one of the variants and commonly used (Neal 2011), is employed in this work. Hamiltonian dynamics are simulated for L steps, each with a step size ε. These are the two adjustable parameters, that determine the efficiency of the HMC algorithm. The leapfrog integrator does not conserve the Hamiltonian exactly and, hence, a Metropolis accept-reject step needs to be performed to ensure invariance of the joint posterior density $p(\theta, p)$. The gradient of potential energy required for the computation of Hamiltonian dynamics, is evaluated using adjoint method.

4. Numerical Implementation and Results

In this section, the performance of the above-described inversion method is investigated through a synthetic ERT data. The inversion problem is to determine the spatial variation of the EC in a domain. For this, firstly, a synthetic EC profile is created in a domain (300 m wide and 150 m deep) as shown in Figure 1a. The EC is gradually varying from about 0.2 S/m in a region near the surface and reducing to about 0.02 S/m towards the bottom and lateral boundaries. This EC field is generated on a fine mesh (Figure 1a) with a correlation length of 100 m and a scale of 1. The forward model with the fine mesh is used for input data generation.

Pole-pole array is chosen to generate the measurement data (electrical potential). 11 current electrode positions are considered on the ground surface at an interval of 10 m and covering a distance of 100 m. The potential electrodes, which are the measurement points, are positioned at an interval of 5 m in this distance of 100 m. The configuration of source current and measurement electrode is illustrated in Figure 1b. Thus, there are 20 measurement points for each current electrode and 220 measurements in total. This measurement data is added with a Gaussian noise with a mean value of 0 and standard deviation of 10% of the corresponding measurement value. This becomes the final measurement values used as input for inversion.

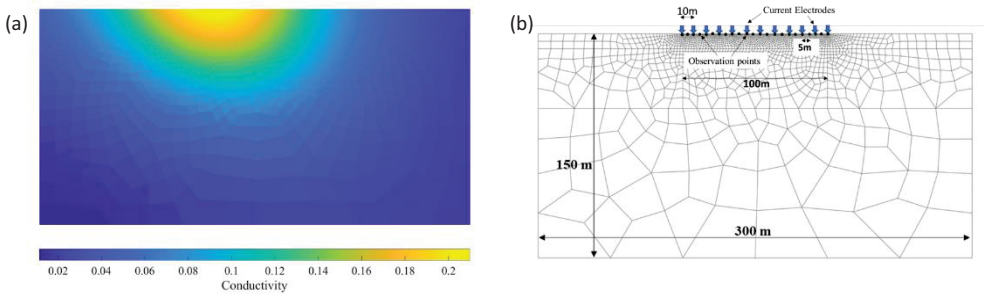


Figure 1a: True EC field; 1b: Coarse mesh used in inversion and, source and observation locations

The domain size, correlation length and scale as considered for generation of measurement data, is taken for inversion. A coarse mesh with 2134 elements (one fourth of fine mesh) is used in inversion. For the considered correlation length and scale, 7 KL terms are chosen based on the decay of the eigen values of the covariance matrix in the KL expansion, thus considerably reducing the number of unknown parameters. A total of 10,000 samples are generated using HMC and the number of leapfrog steps in HMC are selected from a Gaussian i.e. $L \sim N(10,1)$. The step size ε is set automatically in such a manner that the adaptation vanishes gradually as more samples are drawn from the posterior. In this work, a variant (Hoffman and Gelman, 2011) of the original dual averaging scheme is used. The inversion algorithm is run for two different initial EC field (Figure 2).

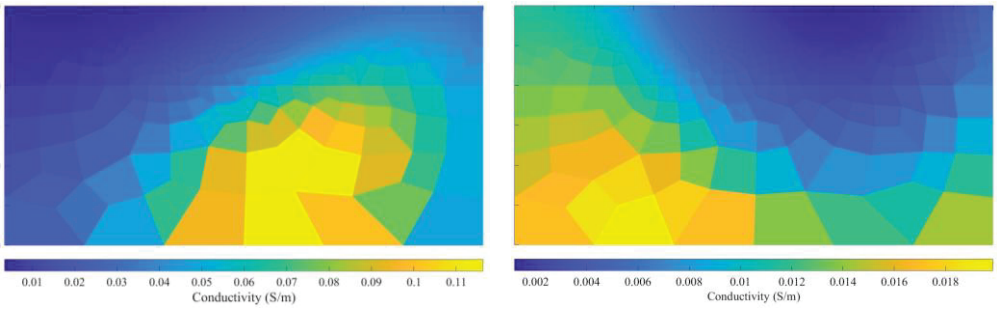


Figure 2: Initial EC field for the two runs of inversion

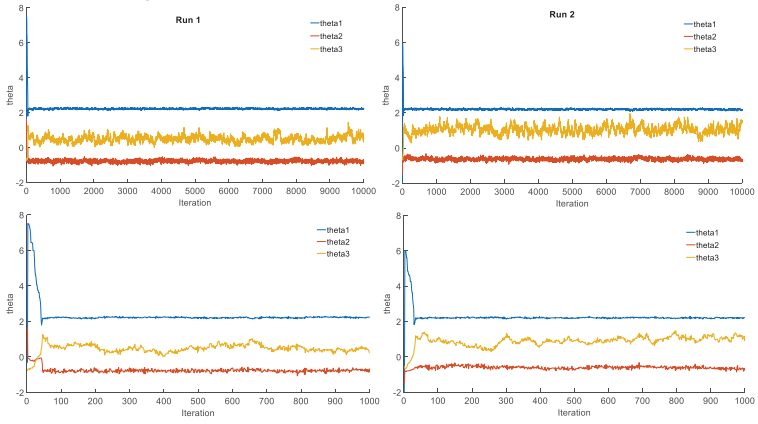


Figure 3: Plot of variation of parameters (θ) with iteration

The Markov chains for the first three truncated KL expansion parameters θ_1, θ_2 and θ_3, that have maximum contribution to the electrical potential, are presented for the two runs, as shown in Figure 3. From the top two plots, it can be confirmed that the Markov chains are stable. From the bottom two plots, it can be seen that the transition from initial state to stationary state is achieved in less than 100 iterations in both the runs. First 200 samples are considered as warm-up samples and are discarded for all post processing calculations.

The 2.5, 50 (median) and 97.5 percentiles are evaluated from the HMC samples for every element and plotted as EC fields, as shown in Figure 4. The 2.5/97.5 percentiles represent the 95% credibility intervals that represent the bounds within which the estimated EC may vary. The true EC considered to generate the measurement data, is found to lie within the 95 % credibility interval and the median value almost matches with the true EC. A good match was found for the second run as well. An idea of the statistical distribution of the identified parameters is obtained by plotting two dimensional marginal distributions between the first four parameters, as shown in Figure 5. Large correlations observed between parameters $\theta_3 - \theta_4$ warrant the use of sophisticated sampling algorithms such as HMC for inversion of ERT data.

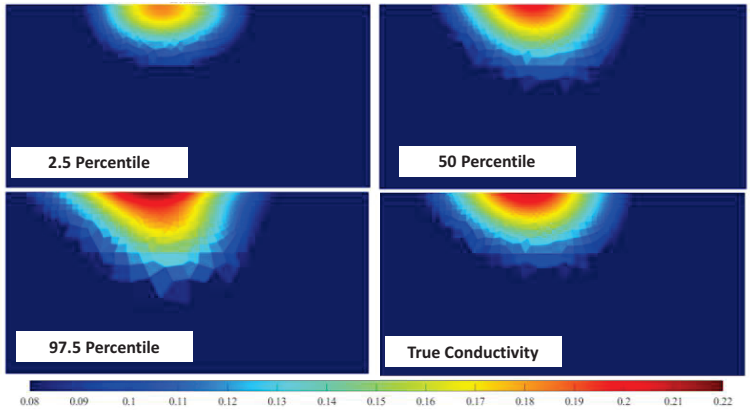


Figure 4: Plot of 2.5, 50, 97.5 percentiles and true EC fields for run1

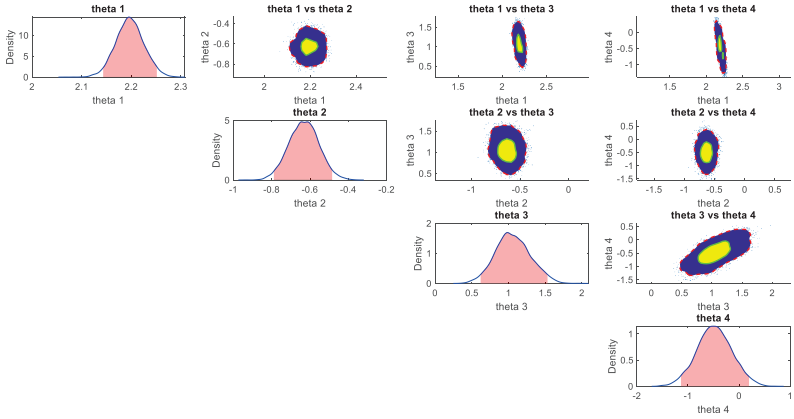


Figure 5: 95% and 50% HPD regions for 1D and 2D marginal distributions for run1

5. Conclusion

In this work an alternative approach for probabilistic inversion of ERT data was proposed using the Hamiltonian Monte Carlo technique, an MCMC method. HMC utilizes the gradient of the posterior for Hamiltonian dynamics and thus can propose samples far from the initial state. The probabilistic inversion using HMC allowed uncertainty appraisal of the estimated parameters. The computational effort was decreased considerably by reducing the dimensionality of the problem with the aid of KL expansion method. The forward modelling of ERT required for the inversion was also developed. The reliability of the approach was demonstrated through inversion of a synthetic problem. The true electrical conductivity value was well within the 95% credibility interval and also matched well with the estimated median value. The authors intend to apply the presented approach to real-world ERT data which will be challenging since the ground surface may be undulating and there may be abrupt changes in the EC profile which may not be accurately captured due the use of Gaussian priors.

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