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EFFICIENT AND ROBUST METHOD FOR RELIABILITY ANALYSIS OF GEOTECHNICAL ULTIMATE LIMIT STATES

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This paper presents the Hyperspheric Integral Reliability Method (HINT) as an efficient and robust alternative for reliability analysis of geotechnical ultimate limit states. HINT is based on the mechanism of the shear strength reduction method, which is available in major geotechnical finite element method software. HINT utilizes the observation that the factor of safety is not only a pointwise estimate of safety, but also a measure of distance to the failure limit along a radial direction. This observation is used to transform the reliability integral to the hyperspheric coordinate system and formulate computationally efficient estimator of failure probability for geotechnical ultimate limit states. The performance of the method was examined on a practical problem of a shallow foundation on sand subjected to vertical load. HINT demonstrated efficient and robust performance in comparison to the Monte Carlo method.

Keywords: Reliability, geotechnical,ultimate, failure, probability, hyperspheric.

1. Introduction

This paper examines the application of the Hyperspheric Integral Reliability Method (HINT) (Depina 2025) on reliability analyses of geotechnical ultimate limit states. The development of the HINT method was motivated by the Shear Strength Reduction Method (SSRM), which is available in major geotechnical finite element method software (e.g., Bentley 2024). The SSRM evaluates the Factor of Safety, F_S, of a geotechnical design by gradually reducing the shear strength parameters until failure is reached. F_S is then defined as the ratio of the initial values and the values resulting in failure:

$$F_S = \frac{c_i}{c_f} = \frac{\tan\phi_{initial}}{\tan\phi_{failure}} = \frac{s_{u,initial}}{s_{u,failure}} = \frac{t_{initial}}{t_{failure}} \quad (1)$$

where$c$ is cohesion, $\tan\phi$ is tangent of friction angle, s_u is undrained shear strength, and t is tensile strength. The SSRM is implemented by applying the same multiplier, $1/F_S$, to all of the shear strength parameters, which means that the SSRM is a line search method that is finding the failure limit along a radial line passing through the initial shear strength parameter values and the origin of the space of the shear strength parameters, $\boldsymbol{X} = [X_1, \dots, X_n]^{\boldsymbol{T}} \in \boldsymbol{D}$,as shown in Fig. 1 (a). The SSRM is sometimes applied only to a subset of the shear strength parameters, $\boldsymbol{P} \subset \boldsymbol{D}$,to exclude some shear strength parameters in SSRM analyses. This is commonly done to exclude certain failure modes in stability analyses (e.g., shallow failure surfaces in slope stability analyses), structures not included in geotechnical failure modes (e.g., steel and concrete structural elements), and other non-strength random variables (e.g., loads, geometry).

To account for these situations, a general vector of random parameters is divided into the parameters being adjusted by the SSRM, $\boldsymbol{P}$, and the parameters excluded from the SSRM, $\boldsymbol{Q}$, such that $\boldsymbol{X} = \boldsymbol{P} \cup \boldsymbol{Q}; \{\boldsymbol{P} = [X_1, \dots, X_r]^T \in \mathbb{R}^r, \boldsymbol{Q} \in [X_{r+1}, \dots, X_n]^T \in \mathbb{R}^s, n = r + s\}$.This is illustrated in Fig.1 (b), where $\boldsymbol{P} = [X_1, X_2]^T$ and $\boldsymbol{Q} = X_3$. From Fig. 1 (b) it can be observed that the SSRM is performed radially in the subset of $\boldsymbol{D}$ spanned by $\boldsymbol{P} = [X_1, X_2]^T \subset \boldsymbol{D}$, from $p_{initial}$ until $p_{failure}$.

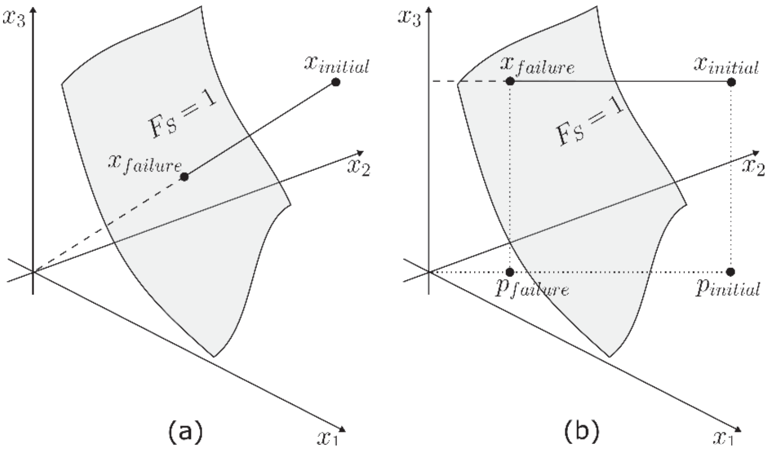


Fig. 1.SSRM with all shear strength parameters being adjusted in (a) and only a subset of parameters, $[x_1, x_2]$, being adjusted in (b).

2. Reliability analysis with HINT

Reliability of a new or existing geotechnical structure is defined as the probability of satisfying a design criterion. Reliability is often expressed in terms of its complement, failure probability, P_F, or the probability of not satisfying a design criterion. Let us consider an n-dimensional vector of random variables, $\boldsymbol{X} \in [X_1, \dots, X_n]^T \in \boldsymbol{D}$, distributed according to the joint probability density function (pdf), $f(\boldsymbol{x})$, where $\boldsymbol{x} \sim f(\boldsymbol{x})$ is a realization of the random vector $\boldsymbol{X}$. P_F is defined as:

$$P_F = P(\boldsymbol{X} \in F) = \int_F f(\boldsymbol{x}) d\boldsymbol{x} = \int_{\mathbb{R}^n} I_F(\boldsymbol{x}) f(\boldsymbol{x}) d\boldsymbol{x} \quad (2)$$

where F is the failure domain, I_F is an indicator function such that $I_F(\boldsymbol{x}) = 1$ if$\boldsymbol{x} \in F$ and $I_F(\boldsymbol{x}) = 0$ otherwise. The indicator function is evaluated based on the outputs of a performance function, $g(\boldsymbol{x})$, which divides $\boldsymbol{D}$ into a safe, $g(\boldsymbol{x}) > 0$, and a failure domain, $g(\boldsymbol{x}) \leq 0$, which are separated by the failure limit.

To account for the possibility of certain parameters being excluded from the SSRM, the extended HINT formulation (Depina 2025) will be presented here. The extended HINT formulation starts by separating the vector of random variables into the ones adjusted by the SSRM and the ones excluded from the SSRM, $\boldsymbol{X} = \boldsymbol{P} \cup \boldsymbol{Q}; \{\boldsymbol{P} = [X_1, \dots, X_r]^T \in \mathbb{R}^r, \boldsymbol{Q} \in [X_{r+1}, \dots, X_n]^T \in \mathbb{R}^s, n = r + s\}$.A limitation of the HINT method is that the uncertainties in the shear strength parameters need to be expressed in term of the parameters adjusted by an SSRM algorithm. The separation of the random variables is also applied to the joint pdf such that $f(\boldsymbol{x}) = f(\boldsymbol{p}|\boldsymbol{q}) \cdot f(\boldsymbol{q})$, where $f(\boldsymbol{p}|\boldsymbol{q})$ is the distribution of $\boldsymbol{p}$ given $\boldsymbol{q}$, and $f(\boldsymbol{q})$ is the marginal distribution of $\boldsymbol{q}$. This allows for the reliability integral in Eq. (2) to be rewritten:

$$P_F = \int_{\mathbb{R}^s} \int_{\mathbb{R}^r} I_F(\boldsymbol{p}, \boldsymbol{q}) f(\boldsymbol{p}, \boldsymbol{q}) d\boldsymbol{p} d\boldsymbol{q} = \int_{\mathbb{R}^s} \left(\int_{\mathbb{R}^r} I_F(\boldsymbol{p}, \boldsymbol{q}) f(\boldsymbol{p}|\boldsymbol{q}) d\boldsymbol{p} \right) f(\boldsymbol{q}) d\boldsymbol{q} \quad (3)$$

HINT exploits the mechanism of the SSRM, in which a radial search is performed in $\boldsymbol{P}$ to locate the failure limit state, by transforming the random variables in P to the high-dimensional spherical or hyperspherical coordinate system:

$$\boldsymbol{S}_P = [r, \theta_1, \dots, \theta_{r-1}]^T = [r, \boldsymbol{\theta}]^T \quad (4)$$

where S_P is a set of hyperspherical coordinates consisting of a radial coordinate, $r \in \mathbb{R}^+$, and $r - 1$ angular coordinates, $\theta = [\theta_1, \dots, \theta_{r-2}, \theta_{r-1}]^T \in [0, \pi] \times \dots \times [0, \pi] \times [0, 2\pi) = \Gamma_P$. By applying the transformation to the reliability integral in Eq. 3, the following expression is obtained:

$$\begin{aligned} P_F &= \int_{\mathbb{R}^s} \left[\int_{\Gamma_P} \left(\int_{\mathbb{R}^+} I_F(r, \boldsymbol{\theta}, \boldsymbol{q}) f(r|\boldsymbol{\theta}, \boldsymbol{q}) k dr \right) f(\boldsymbol{\theta}|\boldsymbol{q}) d\boldsymbol{\theta} \right] f(\boldsymbol{q}) d\boldsymbol{q} \\ &= \int_{\mathbb{R}^s} \int_{\Gamma_P} \left(\int_{\mathbb{R}^+} I_F(r, \boldsymbol{\theta}, \boldsymbol{q}) f(r|\boldsymbol{\theta}, \boldsymbol{q}) k dr \right) f(\boldsymbol{\theta}, \boldsymbol{q}) d\boldsymbol{\theta} d\boldsymbol{q} \\ &= \int_{\mathbb{R}^s} \int_{\Gamma_P} \mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q}) f(\boldsymbol{\theta}, \boldsymbol{q}) d\boldsymbol{\theta} d\boldsymbol{q} = E_{\boldsymbol{\theta}, \boldsymbol{q}}[\mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q})] \quad (5) \end{aligned}$$

where $\mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q}) = \int_{\mathbb{R}^+} I_F(r, \boldsymbol{\theta}, \boldsymbol{q}) f(r|\boldsymbol{\theta}, \boldsymbol{q}) k dr$ is a one-dimensional integral along a radial direction, defined as a function of $\boldsymbol{\theta}$ and $\boldsymbol{q}$, $dV = k dr d\boldsymbol{\theta}$ is the volume element in hyperspherical coordinates, and $E_{\boldsymbol{\theta}, \boldsymbol{q}}$ is the expectation operator. The HINT method with the transformation to the hyperspheric coordinates and the one-dimensional integral along a radial direction are illustrated in Fig. 2.$\mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q})$ can be computed by using numerical integration techniques without performing additional,often computationally demanding, numerical analyses of a geotechnical model to evaluate the performance function. The numerical integration of $\mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q})$is performed here by replacing the integration in $\mathbb{R}^+$ with an interval $[r_{min}, r_{max}]$ such that the probability content outside of the interval is negligible for the considered integral.

Given a set of independent samples from the pdf, $\{\boldsymbol{x}_i \sim f(\boldsymbol{x}); i = 1, \dots, N\}$, the samples are split into the parameters adjusted by the SSRM, $\{\boldsymbol{p}_i; i = 1, \dots, N\}$ and the parameters excluded from the SSRM, $\{\boldsymbol{q}_i; i = 1, \dots, N\}$.The SSRM is performed for each $\boldsymbol{p}_i$, to evaluate the factor of safety, $\{F_{Si}; i = 1, \dots, N\}$ while the corresponding $\boldsymbol{q}_i$ values are kept constant. $\boldsymbol{p}_i$ values are transformed to the hyperspheric coordinates to evaluate $\mathcal{H}(\boldsymbol{\theta}_i, \boldsymbol{q}_i)$.

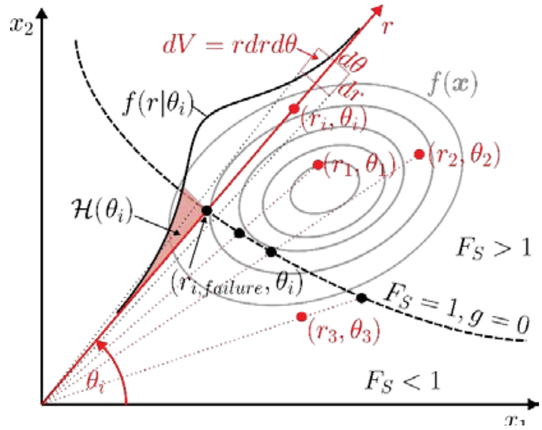


Fig. 2.HINT method and the one-dimensional integral along a radial direction(Depina 2025).

An unbiased Monte Carlo (MC) estimator of P_F in Eq. 5 is evaluated as follows:

$$\hat{P}_F = \frac{1}{N}\sum_{i=1}^{N} \mathcal{H}(\boldsymbol{\theta}_i, \boldsymbol{q}_i) \quad (6)$$

An unbiased estimator of the variance of $\hat{P}_F$can be developed as shown inDepina (2025):

$$Var[\hat{P}_F] = \frac{1}{N}\left(\frac{1}{N-1}\sum_{i=1}^{N} \mathcal{H}(\boldsymbol{\theta}_i, \boldsymbol{q}_i)^2 - \hat{P}_F^2\right) \quad (7)$$

Coefficient of variation of $\hat{P}_F$is estimated as $CoV[\hat{P}_F] = \sqrt{Var[\hat{P}_F]}\,/\hat{P}_F$. The HINT estimator in Eq. 6 improves on the performance of the traditional MC estimator (e.g., Fenton and Griffiths, 2008). This is due to the variance of the HINT estimator depending on the variance of $\mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q})$, as shown in Eq. 7. $\mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q})$ is a continuous random variable with the values in [0, 1], whilethe indicator function, I_F, in the MC estimatoris a discrete random variable taking the values 0 or 1. Since$\mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q}) \leq 1$ it follows that $\mathcal{H}^2(\boldsymbol{\theta}, \boldsymbol{q}) \leq \mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q})$. A relation between the HINT and the MC estimatorscan be developed as shown in Depina (2025):

$$Var[\mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q})] = E[\mathcal{H}^2(\boldsymbol{\theta}, \boldsymbol{q})] - E^2[\mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q})] \leq E[\mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q})] - E^2[\mathcal{H}(\boldsymbol{\theta}, \boldsymbol{q})]$$
$$= P_F - P_F^2 = P_F(1 - P_F) = Var[I_F]$$

where $Var[I_F]$ is the variance of the indicator function in the MC estimator (e.g., Fenton and Griffiths, 2008).This shows that the variance of the HINT estimator will always be equal or smaller than the variance of the MC estimator (Depina, 2025).

3. Bearing capacity of a shallow foundation on sand

The performance of the HINT method is examined on a bearing capacity problem of a shallow foundation on sand modelled in Plaxis 2D (Bentley 2024). The problem features a perfectly rough, rigid foundation with uniformly distributed vertical load on sandy soil. The soil domain is modelled as a rectangular domain with the width of 4.8 m and height of 1.6 m. The rigid foundation is 1 m wide and modelled as a stiff plate without an interface between the plate and the soil that would allow for relative displacements. The soil is modelled as dry with the Mohr Coulomb soil model featuring cohesion, c, and the tangent of the friction angle, $\tan\phi$, as the shear strength parameters. The shear strength parameters are uncertain and constant across the soil domain. c is assumed to be lognormally distributed with the mean of $\mu_c = 6$ kPa and standard deviation of $\sigma_c = 1.2$ kPa. $\tan\phi$is assumed to be normally distributed with the mean of $\mu_{\tan\phi} = \tan 30°$ and standard deviation of $\sigma_{\tan\phi} = 0.1 \cdot \mu_{\tan\phi}$. $c$and $\tan\phi$are assumed to be independent. The load is also uncertain and distributed according to the Gumbel distribution with the mean of $\mu_w = 150$ kPa and the standard deviation of $\sigma_w = 30$ kPa. The geometry of the model with the failure mode for a realization of the random properties is shown in Fig. 3. The model was analysed using 2050 samples of the random parameters to determine the corresponding F_S values. Given sufficient samples, the results of the HINT method can be also employed to estimate the moments and the empirical distribution of F_S.

The distribution of F_S values is shown in Fig. 4 (a) with the mean value of $\mu_{F_S} = 1.27$ and standard deviation of $\sigma_{F_S} = 0.16$.

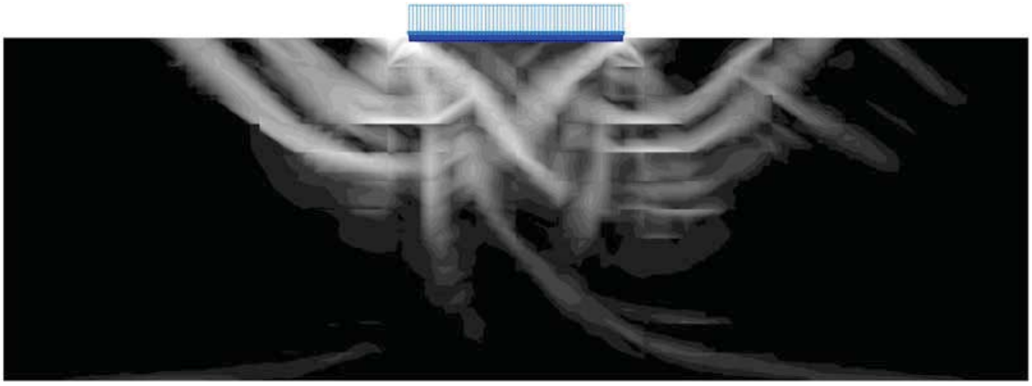
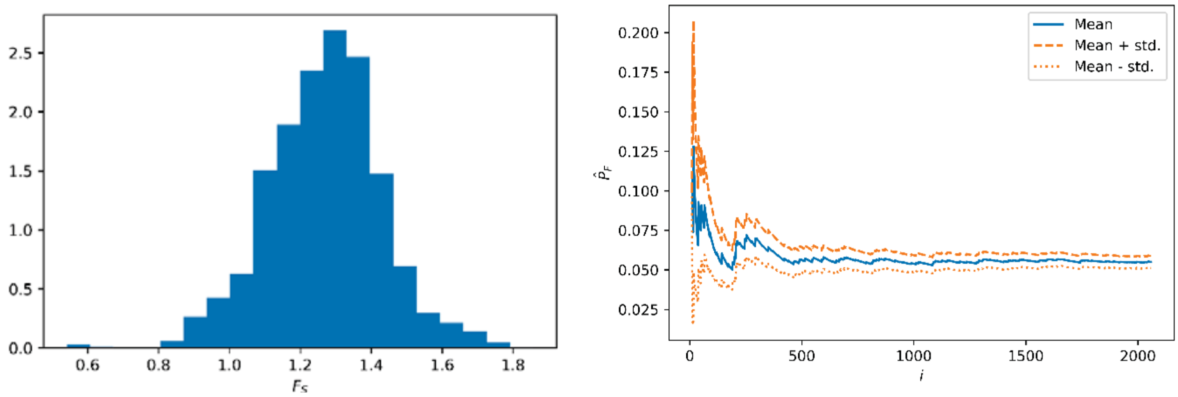


Fig. 3.Failure mode illustrated by incremental shear strains after the SSRM analysis in Plaxis 2D.

$\hat{P}_F$ was estimated with the HINT method on the 2050 samples to be $\hat{P}_F = 5.505 \cdot 10^{-2}$ with the coefficient of variation of $CoV[\hat{P}_F] = 6.804 \cdot 10^{-2}$.An estimate of $\hat{P}_F$ was also made with the MC method on the same samples with $\hat{P}_{F,MC} = 5.20 \cdot 10^{-2}$ and $CoV[\hat{P}_{F,MC}] = 9.412 \cdot 10^{-2}$. Figure 4 (b) shows the convergence of the $\hat{P}_F$estimate with the HINT method and the associated variability in the estimate, expressed based on the square root of the variance in Eq. 7. The estimates of the failure probability are consistent, with the HINT method showing better convergencein terms of $CoV[\hat{P}_F]$ for the same number of samples. The difference in performance of the two methods is expected to increase for lower failure probability levels and coefficients of variation.



(a) (b)

Fig. 4.HINT method: (a) distribution of F_S values and (b) convergence of the $\hat{P}_F$ estimate.

4. Conclusions

This paper provided an overview of the HINT method and its potential for computationally efficient and robust assessment of failure probabilities for geotechnical ultimate limit states. The HINT method exploits the radial search mechanism in the shear strength reduction method to transform the reliability integral to the hyperspheric coordinate system. This allowed for improvements in the performance of the HINT method in comparison with the classical Monte Carlo method. The performance of the HINT method was examined on a practical problem of bearing capacity of a shallow foundation on sandy soil. HINT method demonstrated efficient and robust performancein comparison with the Monte Carlo method.

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