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Three-Dimensional Random Field Modeling of Soil Properties Considering Cross-Correlation

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Abstract: It is widely recognized that there is spatial variability in soil properties in the geotechnical engineering. Spatially variable soils demonstrate auto-correlation of parameters, and also show significant cross-correlation between different parameters. At present, random field theory is often used to model the spatial variability of soil. However, in most studies, only two-dimensional (2D) random field is established, and there is a limitation of research on establishing three-dimensional (3D) random field. This paper aims to develop a 3D random field modeling method based on modified matrix decomposition method (M-MD) characterize the soil heterogeneity in 3D. Compared with the traditional matrix decomposition method (MD), the modified matrix decomposition considers the influence of relative distance on the cross-correlation matrix, which can more accurately reflect the cross-correlation between different soil parameters. The proposed method is applied to generate a multi-parameter 3D random field model considering cross-correlation between cohesion and friction angle. This study paves a way for future safety and risk assessment considering cross-correlation in geotechnical engineering.

Keywords: Spatial variability; 3D Random field; Modified matrix decomposition method; Cross-correlation structure.

1. Introduction

As a product of nature, the inherent soil variability has been widely recognized as one of the most important key factors of the uncertainty in geotechnical engineering (Phoon and Kulhawy 1999; Jiang and Huang 2016; Cho 2007). The description of the uncertainty of soil properties has become a key scientific problem that researchers have been committed to solving. In the past, the random variables were used to characterize the uncertainty of soil parameters. Although this method reflects the spatial randomness of soil properties, it ignores the spatial correlation (Chen et al. 2016). According to the relevant studies, the spatially variable soils demonstrate auto-correlation of parameters, and also show significant cross-correlation between different properties (Yucemen et al. 1973; Chen et al. 2016; Huang et al. 2021).

Since Vanmarcke (1984) put forward the random field theory, it has been widely used in geotechnical engineering risk assessment. The commonly used random field methods include local average subdivision (LAS) method (Hicks et al. 2008; Fenton and Vanmarcke, 1990), K-L expansion method (Cho, 2010; Jiang et al. 2014; Al-Bittar and Soubar, 2013), fast Fourier transform (FFT) method (Oliver 1995; Li et al. 2021) and matrix decomposition (MD) method (Li et al. 2015; Cheng et al. 2019; Huang et al. 2021). Among these methods, the MD method based on standard Cholesky algorithm has the advantages of simple calculation process and easy programming, which is widely used in geotechnical engineering. However, in the literature, most of the established random fields are confined to two-dimensional condition (Jiang et al. 2014; Yang et al. 2019; Tian et al. 2021a,b). There is a limitation of research on establishing three-dimensional random field.

The objective of this study is to develop a modified matrix decomposition (M-MD) method to establish the 3D multi-parameter random field with cross-correlated. Compared with MD method, M-MD method considers the influence of spatial location information when establishing cross-correlation matrix. To better illustrate the developed method, the M-MD method is applied to generate a two-parameter random field of soil shear strength $(c - \varphi)$ considering negative correlation. The correlation statistics of the established random field is also carried out to validate its rationality and accuracy.

2. Three-dimensional random field modeling

2.1 Mathematical model of M-MD method

Herein, it will mainly introduce the construction of the mathematical model and the solution process of the M-MD method. As aforementioned, there is an auto-correlation between soil properties. Figure 1 shows the 2-D region discrete distribution containing $m \times n$ spatial points. For two points i and j in the region, the auto-correlation information of the two points can be expressed as an auto-correlation coefficient ρ_{ij}. By combining the auto-correlation coefficient of all spatial points in the region, the auto-correlation matrix reflecting the auto-correlation information between point and point in the region can be expresses as:

$$(\sum_{\xi;\xi}^{k,G})_{n_e\times n_e}=\begin{bmatrix}\rho_{11} & \rho_{12} & \cdots & \rho_{1n_e}\\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n_e}\\ \vdots & \vdots & \ddots & \vdots\\ \rho_{n_e1} & \rho_{n_e2} & \cdots & \rho_{n_en_e}\end{bmatrix}_{n_e\times n_e} \tag{1}$$

where the $n_e = m \times n$, and the $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}$ is the auto-correlation matrix of the region with $n_e \times n_e$ dimensional respectively. As the physical meaning of the auto-correlation represents, the closer the distance between two spatial points, the stronger the auto-correlation of the soil properties. With the increase of the distance of the two spatial points, the auto-correlation gradually decreases, leading to a decrease of the auto-correlation coefficient. When the two spatial points have the same coordinates, the soil properties have the strongest auto-correlation and the auto-correlation coefficient reaches the maximum (i.e., 1). After establishing the auto-correlation matrix $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}$ of all the spatial points in region, we can construct the mathematical model of M-MD method.

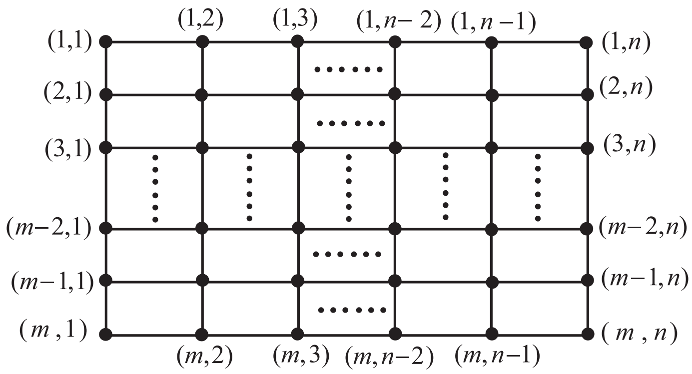


Figure 1. Region discrete distribution

Regarding the random field as a random vector, according to the theory of random vector, the correlation between different vectors reflects the covariance of any two random vectors. Thus, from the perspective of linear algebra, matrix transformation is a feasible inversion method. Since the non-Gaussian distribution can be obtained by equal probability transformation of Gaussian distribution, for the convenience of discussion, it is assumed that all parameters subject to Gaussian distribution. Taking the single parameter $H_1^{k,G}$ as an example, the mathematical inversion equation can be constructed as follows:

$$H_1^{k,G}=L_{11}\xi_1 \tag{2}$$

where L_{11} represents the matrix to be determined, and the ξ_1 is a vector of independent standard normal random samples, respectively. Through the MD method, the auto-correlation matrix of $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{11}$ is:

$$(\sum_{\xi;\xi}^{k,G})_{n_e\times n_e}^{11}=\mathrm{Cov}(H_1^{k,G},H_1^{k,G})=L_{11}L_{11}^{T} \tag{3}$$

According to Eq. (1), it can be found that the auto-correlation matrix $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{11}$ is a real symmetric positive-define matrix. Therefore, the undetermined matrix L_{11} in Eq. (2) can be obtained by Cholesky decomposition method. Then, the random field of $H_1^{k,G}$ can be sufficiently defined. Hence, for a single parameter random field, the auto-correlation performance can completely determine the inversion equation.

For multi-parameter random fields, the cross-correlation between different parameters is also a key factor to be considered. Taking the two parameters $H_1^{k,G}$ and $H_2^{k,G}$ random field modeling as an example, the mathematical inversion model needs to meet not only the auto-correlation of the $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{11}$ and the $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{22}$, but also the cross-correlation function of $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{12}$. The inversion equation of parameter $H_1^{k,G}$ can still be constructed by Eq. (2). Since a complete correlation is established between $H_1^{k,G}$ and ξ_1, $H_2^{k,G}$ must also establish a cross-correlation with $H_1^{k,G}$ through ξ_1. However, the two parameters $H_1^{k,G}$ and $H_2^{k,G}$ are not completely cross-correlated generally. That is, the cross-correlation matrix $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{12}$ is not equal to its auto-correlation matrix $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{22}$. For example, the principal diagonal element of the auto-correlation matrix $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{22}$ is usually 1, while the principal diagonal element of cross-correlation matrix $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{12}$ is often equal to a coefficient between - 1 and 1. Thus, if the inversion equation of parameter $H_2^{k,G}$ is constructed as $H_2^{k,G} = L_{12}\xi_1$, it does not satisfy its own auto-correlation. At this time, we consider adding another term to Eq. (4) to meet its own auto-correlation. Based on this idea, the mathematical inversion model of $H_2^{k,G}$ can be constructed as:

$$H_2^{k,G} = L_{12}\xi_1 + L_{22}\xi_2 \tag{4}$$

where ξ_2 is a vector of independent standard normal random samples, L_{12} and L_{22} are the undetermined matrices to be solved. The cross-correlation between ξ_1 and ξ_2 is 0. According to the real symmetry of auto-correlation matrix and cross-correlation matrix, there is a relationship as follows:

$$(\sum_{\xi;\xi}^{k,G})_{n_e\times n_e}^{11} = L_{11}L_{11}^T \tag{5}$$

$$(\sum_{\xi;\xi}^{k,G})_{n_e\times n_e}^{12} = L_{12}L_{11}^T \tag{6}$$

$$(\sum_{\xi;\xi}^{k,G})_{n_e\times n_e}^{22} = L_{12}L_{12}^T + L_{22}L_{22}^T \tag{7}$$

According to Eq. (5-7), the undetermined matrices L_{11}, L_{12} and L_{22} in the two parameters random fields inversion model of Eq. (2) and Eq. (4) can be solved. By analogy, a multi-parameter random field inversion model can be constructed.

2.2 Matrix solution of M-MD method

This section aims to introduce the flow path of generating the random field using M-MD method. For two parameters $H_1^{k,G}$ and $H_2^{k,G}$, the spatial geometric characteristics are determined before establishing the random field model. For the discrete region in three-dimensional, the auto-correlation of spatial points can be calculated directly according to the theoretical correlation function. There are five commonly used auto-correlation functions, including single exponential, squared exponential, second-order Markov, cosine exponential and the binary noise. The exponential function is the most used to establishing random field in geostatistical analysis. This paper using the single exponential to calculate the auto-correlation matrix. The single exponential function is expressed as:

$$\rho(\tau_x, \tau_y, \tau_z) = \exp\left[-2\times(\frac{\tau_x}{\theta_x} + \frac{\tau_y}{\theta_y} + \frac{\tau_z}{\theta_z})\right] \tag{8}$$

where τ and θ is the relative distance between different spatial points and the auto-correlation distance in the corresponding direction, respectively. Through Eq. (1) and Eq. (9), it is interesting to find that the correlation function is only related to the relative distance τ of spatial points in the region. In other words, for a discrete region, the auto-correlation matrices of all parameters are the same. The auto-correlation matrices $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{11}$ and $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{22}$ can be solved by Eq. (1) and Eq. (8). The cross-correlation matrix $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{12}$ is obtained by multiplying $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{11}$ and a cross-correlation coefficient χ_{12}.

The undetermined matrix L_{11} can be obtained by Cholesky decomposition of $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{11}$, i.e.,

$$L_{11} = \text{Cholesky}((\sum_{\xi;\xi}^{k,G})_{n_e\times n_e}^{11}) \tag{9}$$

From Eq. (6), the following relationship can be deduced:

$$L_{12} = (\sum_{\xi;\xi}^{k,G})_{n_e\times n_e}^{12} (L_{11}^T)^{-1} \tag{10}$$

According to $(\sum_{\xi,\xi}^{k,G})_{n_e\times n_e}^{11} = L_{11}L_{11}^T$ and Eq. (10), it can be concluded that

$$L_{12} = (\sum\nolimits_{\xi;\xi}^{k,G})^{12}_{n_e \times n_e} (L_{11}^{-1} (\sum\nolimits_{\xi;\xi}^{k,G})^{11}_{n_e \times n_e})^{-1} = (\sum\nolimits_{\xi;\xi}^{k,G})^{12}_{n_e \times n_e} (\sum\nolimits_{\xi;\xi}^{k,G})^{11\ -1}_{n_e \times n_e} L_{11} \tag{11}$$

The remaining auto-correlation of parameter $H_2^{k,G}$ is $(\sum_{\xi,\xi}^{k,G})^{22}_{n_e \times n_e} - L_{12}L_{12}^T$. Considering the $(\sum_{\xi,\xi}^{k,G})^{22}_{n_e \times n_e}$ and $L_{12}L_{12}^T$ are the real symmetric matrices, the undetermined matrix L_{22} can be solved by Cholesky decomposition of remaining auto-correlation matrix which can be expressed as:

$$L_{22} = \text{Cholesky}((\sum\nolimits_{\xi;\xi}^{k,G})^{22}_{n_e \times n_e} - L_{12}L_{12}^T) \tag{12}$$

When building a three-dimensional cross-correlation random field, the cross-correlation matrix is expressed as:

$$(\sum\nolimits_{\xi;\xi}^{k,G})^{12}_{n_e \times n_e} = \chi_{12} \begin{bmatrix} 1 & \rho(\tau_{x12}, \tau_{y12}, \tau_{z12}) & \cdots & \rho(\tau_{x1n_e}, \tau_{y1n_e}, \tau_{z1n_e}) \\ \rho(\tau_{x21}, \tau_{y21}, \tau_{z21}) & 1 & \cdots & \rho(\tau_{x2n_e}, \tau_{y2n_e}, \tau_{z2n_e}) \\ \vdots & \vdots & \ddots & \vdots \\ \rho(\tau_{xn_e1}, \tau_{yn_e1}, \tau_{zn_e1}) & \rho(\tau_{xn_e2}, \tau_{yn_e2}, \tau_{zn_e2}) & \cdots & 1 \end{bmatrix} \tag{13}$$

where the χ_{12} is the cross-correlation coefficient of the properties, and the $\rho_{\tau_{xij}}$, $\rho_{\tau_{yij}}$, $\rho_{\tau_{zij}}$ is the correlation coefficient between the ith and jth point in x-, y-, z-direction respectively. Then, the corresponding matrices L_{11}, L_{12}, and L_{22} are solved by Eq. (9), Eq. (11) and Eq. (12). The pseudo-random number generator is used to generate n_e independent random variables ξ_1 and ξ_2 which subjecting to Gaussian distribution. Finally, all the undetermined matrices are solved by above steps and the two-parameter random fields are established by Eq. (2) and Eq. (4).

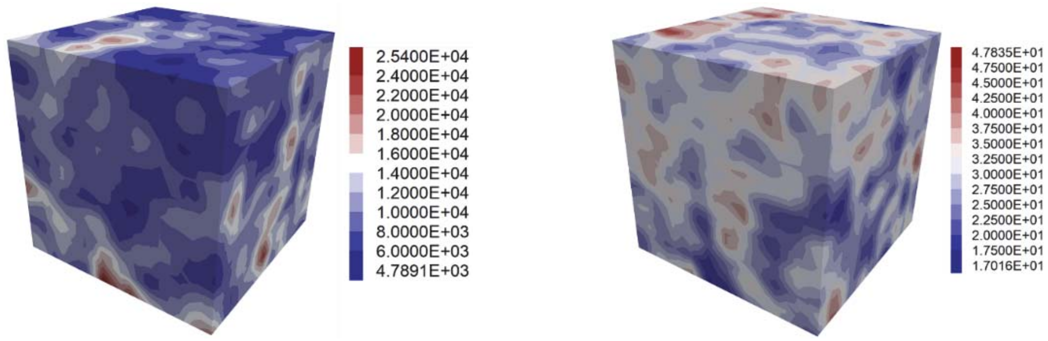
3. Illustration example

3.1 Generation of 3-D cross-correlation random fields of cohesion and friction angle

In order to test the rationality of the proposed method, 100 random fields are generated by M-MD method, and then the statistical results are used for correlation statistics. The applicability of the M-MD method is inversely verified through the comparison between the statistical results and the preset auto-correlation function. As an important strength index of soil properties, shear strength is of great significance in slope reliability and tunnel face stability. What's more, after the investigation of a large number of research, it is found that there is a negative correlation between cohesion and friction angle. Thus, this paper takes the soil shear strength as example to generate the cross-correlated random field. The single exponential function is used as the theoretical auto-correlation function. Only the isotropy correlation structure is taking in consideration. That is to say, in Eq. (8), the auto-correlation distance $\theta_x = \theta_y = \theta_z = \theta = 2\text{m}$ in this study. Moreover, a 3D simulated region with $20\text{m} \times 20\text{m} \times 20\text{m}$, and the size of unit element with 1m is made herein. The cross-correlation coefficient $\chi_{c\varphi} = -0.7$ between cohesion and friction angle is set in this paper. The statistical characteristics of cohesion and friction angle are listed in Table 1

Table 1. Statistical properties of soil parameters

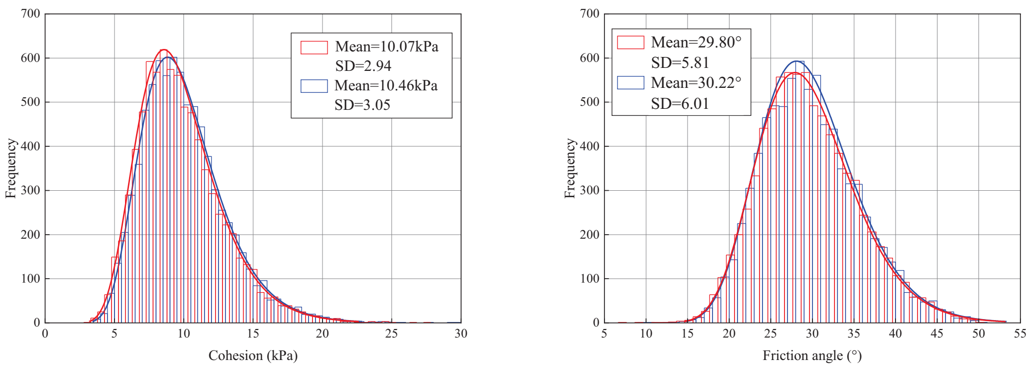
parameters	Mean	Standard deviation	Distribution	Auto-correlation distance	Cross-correlation coefficient
Cohesion (kPa)	10	3	Lognormal	$\theta_x = \theta_y = \theta_z = 2\text{m}$	$\chi_{c\varphi} = -0.7$
Friction angle (°)	30	6	Lognormal	$\theta_x = \theta_y = \theta_z = 2\text{m}$	-



(a) Cohesion (b) Friction angle

Figure 2. Typical realization of 3D cross-correlated random fields between cohesion and friction angle

Figure 2 shows one typical realization of the random field between cohesion and friction angle. It can be seen that it has a well auto-correlation in cohesion and friction angle random field. The greater the cohesion, the smaller the friction angle, which reflects the negative correlation between random fields. Figure 3 shows two typical realization of frequency distribution histogram of cohesion and friction angle respectively. It can be seen that the generated random field well subjects to the lognormal distribution. For example, among one of the typical realizations of cohesion's random field, the mean of cohesion is 10.07kPa which has an error of 0.695% with the preset value. The standard deviation has an error of 2% with preset value. That is, the simulated value is well fit with the theoretical value. The pattern of friction angle is similar to the cohesion, and also subjects to lognormal distribution.



(a) Cohesion (b) Friction angle

Figure 3. Two typical realization of frequency distribution histogram

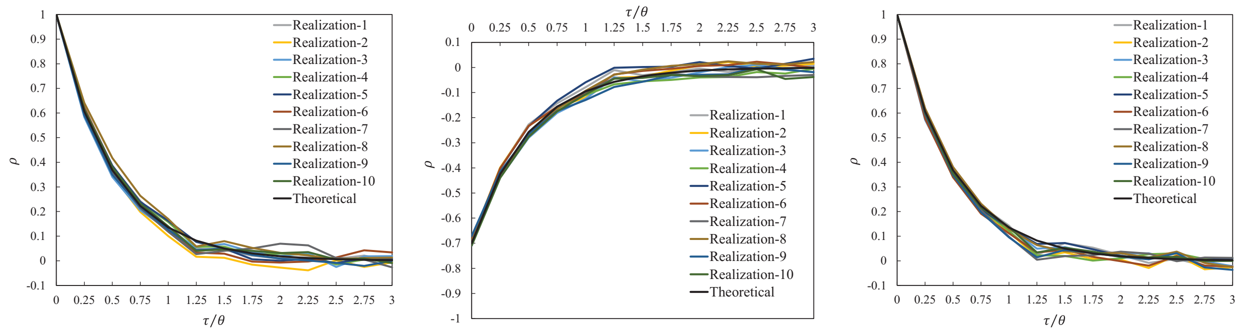
3.2 Statistical results

In order to describe the correlation more intuitively between random fields, the result statistics are carried out in this section. 10 cross-correlated random fields of cohesion and friction angle are stochastically generated by M-MD method. The statistical results of mean and standard deviation are listed in Table 2 It can be seen that the mean and standard deviation of cohesion are well regressed to 10 and 3, and the mean and standard deviation of friction angle are well regressed to 30 and 6, respectively. Although there are some errors which are all within 1%. In the actual calculation process, the Monte-Carlo strategy is often used to make hundreds of operations, and this error will be reduced to a lower level. Therefore, an error of 1% is considered acceptable over here.

Table 2. Statistical results of mean and standard deviation

Parameters	N_P	Mean	Mean of mean	Error	Standard deviation	Mena of standard deviation	Error
	1	10.0701			2.9393		
	2	9.8746			2.8242		
Cohesion (kPa)	3	10.1559	9.9891	0.109%	3.0008	2.9707	0.967%
	⋮	⋮			⋮		
	100	9.6458			3.0560		
	1	29.8079			5.8048		
	2	29.7966			5.8596		
Friction angle (°)	3	30.0591	29.8851	0.383%	6.0155	5.9862	0.23%
	⋮	⋮			⋮		
	100	31.0230			6.2775		

Figure 4 shows 10 statistical and theoretical curves of correlation of cohesion and friction angle. It can be found that the cohesion random field auto-correlation curve generated based on M-MD method fits well with the theoretical curve. From Figure 4(a) and Figure 4(c), it is easy to find that the auto-correlation of cohesion and friction angle is positively correlated, and decreases exponentially with the increase of distance. The correlation is strong When $\tau \leq 2\theta$, and the theoretical value is 0.018. When $\tau > 2\theta$, the correlation is infinitely close to 0 from 1. When $\tau = 3\theta$, the theoretical correlation is 0.0025. At this time, it is considered that the two spatial points are no longer correlated. As shown in Figure 4(b), the cross-correlation between cohesion and friction angle is negatively correlated. The cross-correlation is gradually close to 0 from -0.7. Similarly, the cross-correlation is regarded as non-existent when $\tau > 3\theta$.



(a) Auto-correlation of cohesion (b) Cross-correlation (c) Auto-correlation of friction angle

Figure 4. Statistical and theoretical curves of correlation

4. Conclusion

Aiming at the limitation of less research on three-dimensional random field, this study developed a modified matrix decomposition (M-MD) method to generate 3D multi-parameter random fields with cross-correlation based on Cholesky algorithm. Taking two-parameter random fields as example, the 3D random field with cross-correlated of soil shear strength is established and the M-MD method is validated. The main conclusions can be drawn from this article as follows:

(1) An essential difference between M-MD method and MD method is that the spatial location information of spatial points is considered when establishing the cross-correlation matrix. The advantage of M-MD is that the cross-correlation between different parameters can be more precisely. This improvement increases the accuracy of the random field to a certain extent.

(2) According to the proposed M-MD method, a 3D random field of cohesion and friction angle considering negative correlation is established. Through the correlation validation, it is easy to find that the error is within 1%, and the generated random field can well reflect the correlation between the cohesion and friction angle. Statistical results show that the correlation is strong when $\tau \leq 2\theta$. It can be considered that the correlation nearly disappeared when $\tau > 3\theta$.

(3) The establishment of 3D random field will be limited by the performance of computer. For example, a typical computer with 8GB of memory can only theoretically generate a random field with 10000 discrete spatial points. The performance of computer limits the development of 3D random fields. Thus, it is necessary to develop a high-performance random field generation method based on M-MD method in the further research stage.

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