Proc. of the 8th International Symposium on Geotechnical Safety and Risk (ISGSR)
Edited by Jinsong Huang, D.V. Griffiths, Shui-Hua Jiang, Anna Giacomini and Richard Kelly
 Published by Research Publishing, Singapore.
doi: 10.3850/978-981-18-5182-7_20-002-cd

ISGSR 2022

An Efficient Method for the Discretization of 3-D Random Fields of Soil Properties in the Stochastic Finite Element Analysis of Geotechnical Problems

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Abstract: The spatial variability of soil properties is a significant aspect that should be considered in the analysis related to geotechnical safety and risk. Random field theory has been used for the discretization of soil properties in space and integrated with the stochastic finite element method for the probabilistic analysis of geotechnical structures. However, efficient discretization of three-dimensional random fields with large geometric size and high definition remains a challenging issue because of the heavy computational costs during the process stemming from the large physical memory demand for the storage of the autocorrelation matrix and the long computing time for the large matrix decomposition. A decomposed Karhunen-Loève expansion scheme was proposed in the present study. The proposed method is applicable when a separable autocorrelation function is employed. In this scheme, the generation of a three-dimensional random field will be decomposed into that of three separate one-dimensional random fields, and the eigenpairs needed for the random field discretization could be solved using the autocorrelation matrix in each direction respectively. A stepwise procedure was then employed to further reduce the memory usage when multiplying these one-dimensional solutions to get the final results. The precision and efficiency of the decomposed K-L expansion method for the discretization of random fields are verified. Compared with the traditional method, the proposed method significantly reduces the computing time and storage space, making the discretization of three-dimensional random fields more efficient.

Keywords: random field; spatial variability; soil properties; Karhunen-Loève expansion; stochastic finite element method.

1 Introduction

The spatial variability of soil properties in nature due to load history and geological process has attracted the attention of researchers (Phoon and Kulhawy, 1999a, 1999b; Jiang et al., 2022). Random field theory is alaways used for the discretization of spatially distributed soil properties, and the discretized soil parameters could then be incorporated into the finite element models for the consideration of soil spatial variability in the analysis of geotechnical structures (Sudret and Der Kiureghian, 2002). The stochastic finite element method (SFEM) has already been used in some geotechnical problems (Griffiths et al., 2009; Zhang et al., 2016), however, most of the previous studies are the application of one-dimensional (1-D) or two-dimensional (2-D) random fields of soil properties. Efficient simulation of three-dimensional (3-D) random fields with large geometric sizes and high definition remains a challenging problem because of the strong demand for physical memory space and computational time during the process (Liu et al., 2014; Xiao et al., 2016).

A decomposed Karhunen-Loève (K-L) expansion scheme for the discretization of 3-D random fields was proposed, which is applicable when a separable autocorrelation function (ACF) is used. In the proposed method, the generation of a 3-D random field will be decomposed into that of three separate 1-D random fields, and the eigen solutions needed for the random field generation could be solved using the autocorrelation matrix in each direction respectively, which makes it much more efficient for the discretization of 3-D random fields compared with the traditional methods.

2 The decomposed K-L expansion method

Random fields of soil properties can be simulated based on the statistical features and the correlation structures of the object properties. The correlation structures in space are described by ACFs and the related correlation lengths. The K-L expansion procedure is one of the most commonly used methods for the discretization of random fields (Li et al., 2008; Betz et al., 2014), which is based on the spectral decomposition of the ACF and discretizes random fields as a set of deterministic functions with random variables (Tsantili and Hristopulos, 2016).

The traditional K-L expansion method is not practical for the generation of 3-D random fields. It is because the solutions of the Fredholm integral equation of the second kind are needed to obtain eigen solutions of ACF in the process. For the eigenvalue problem of a 3-D random field with a huge or refined model, even numerical solutions are difficult to work out (Zheng and Dai, 2017). It is because the heavy computational costs are demand, which stems from the large physical memory used for the storage of the autocorrelation matrix and the long computing time for the large matrix decomposition (Cheng et al., 2019). A decomposed K-L expansion scheme was proposed, which was enlightened by a stepwise covariance matrix decomposition algorithm developed by Li et al. (2019). The eigen solutions of the autocorrelation matrix for a 3-D random field were obtained in the proposed method by firstly decomposing the solution into that of separate 1-D random fields in each dimension, and a stepwise procedure was then used to further reduce the memory usage when multiplying these 1-D solutions to get the final results.

2.1 The procedure of the decomposed scheme

The proposed decomposed K-L expansion method is practicable when a separable ACF is used, which can be expressed as the product of 1-D ACFs in each direction as:

$$\rho(\tau_x, \tau_y, \tau_z) = \rho(\tau_x) \cdot \rho(\tau_y) \cdot \rho(\tau_z) \tag{1}$$

where τ_x, τ_y, and τ_z represent the distances between any two coordinate points along x, y, and z directions, respectively. The most commonly used ACFs, i.e. the single and squared exponential functions, are both separable.

The eigen solutions were calculated by the eigen-decomposition of the autocorrelation matrix. The obtained original eigenvectors and eigenvalues need to be normalized before being used in the K-L expansion. After obtaining the eigen solutions, the K-L expansion used for the discretization of random fields can be operated in MATLAB through the matrix computation:

$$\boldsymbol{H}_{(n\times1)} = \mu \cdot \boldsymbol{I}_{(n\times1)} + \sigma \cdot [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_M]_{(n\times M)} \cdot \begin{bmatrix} \sqrt{\lambda_1} & & & \\ & \sqrt{\lambda_2} & & \\ & & \cdots & \\ & & & \sqrt{\lambda_M} \end{bmatrix}_{(M\times M)} \cdot \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_M \end{bmatrix}_{(M\times1)} \tag{2}$$

where $\boldsymbol{H}$ denotes the array storing the values of the discretized random field; In a 3-D random field, the number of element nodes is $n = n_x \times n_y \times n_z$, where n_x, n_y, and n_z are the numbers of nodes in the x, y, and z directions, respectively; $\boldsymbol{I}$ is the vector with all elements being 1 and the dimension being $n\times1$; μ is the mean value of the parameter and σ is the standard deviation; λ_i and $\boldsymbol{\varphi}_i$ denote the eigenvalues and the eigenvectors of the ACF; ξ_i are a series of independent random coefficients obeying a standard normal distribution; M is the truncated order in the K-L expansion.

The autocorrelation matrix $\boldsymbol{C}$ of a 3-D random field with a huge or refined model is typically very large with the dimension of $n_x n_y n_z \times n_x n_y n_z$, the direct eigen-decomposition of which will demand a large amount of physical memory space that usually outruns the capability of a personal computer. The proposed method circumvents the direct eigen-decomposition of the 3-D autocorrelation matrix and conducts the K-L expansion in each dimension, respectively. When the separable ACF is employed, the corresponding separable autocorrelation matrix can be obtained:

$$\boldsymbol{C} = \boldsymbol{C}_z \otimes \boldsymbol{C}_x \otimes \boldsymbol{C}_y \tag{3}$$

where $\boldsymbol{C}_x$, $\boldsymbol{C}_y$, and $\boldsymbol{C}_z$ denote the autocorrelation matrixes for the disassembled 1-D random fields in each direction. Based on the properties of the Kronecker product, when the autocorrelation matrix is separable, the obtained matrixes for eigen solutions are also separable (Yue et al., 2018). We define $\boldsymbol{V}$ as an $M \times M$ matrix as follows, the elements in the diagonal of which are sorted in descending order and obtained as the values of the square root of the eigenvalues.

$$\boldsymbol{V} = diag([\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_M}]) \tag{4}$$

We define $\boldsymbol{F}$ as the corresponding eigenfunction matrix with the dimension being $n\times M$.

$$\boldsymbol{F} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_M] \tag{5}$$

$\boldsymbol{X}$ denotes a matrix of standard random variables with the dimension being $M\times1$

$$X = [\xi_1, \xi_2, \cdots, \xi_M]^T \tag{6}$$

Then we can decompose the matrixes of eigenpairs as:

$$\begin{cases} V = V_z \otimes V_x \otimes V_y \\ F = F_z \otimes F_x \otimes F_y \end{cases} \tag{7}$$

We define the matrix P as:

$$P = F \cdot V \tag{8}$$

Then the K-L expansion can be expressed as

$$H = \mu \cdot I + \sigma \cdot F \cdot V \cdot X = \mu \cdot I + \sigma \cdot (P_z \otimes P_x \otimes P_y) \cdot X \tag{9}$$

in which:

$$\begin{cases} P_x = F_x \cdot V_x \\ P_y = F_y \cdot V_y \\ P_z = F_z \cdot V_z \end{cases} \tag{10}$$

Therefore, we can obtain the eigen solutions in 3-D space by calculating these for the decomposed 1-D random fields in the *x*, *y*, and *z* directions firstly, and then multiplying them using the Kronecker product. Although the computational cost in solving the eigen solutions of the autocorrelation matrix can be reduced by using the decomposed scheme, the operation between large matrixes with the Kronecker product remains a restriction on the discretization of 3-D random fields. A stepwise procedure suggested by Li et al. (2019) in the covariance matrix decomposition has been modified and employed in our study to supersede the Kronecker product between matrixes. See Li et al. (2019) for more details on this stepwise procedure.

2.2 SFEM integrating the 3-D random field

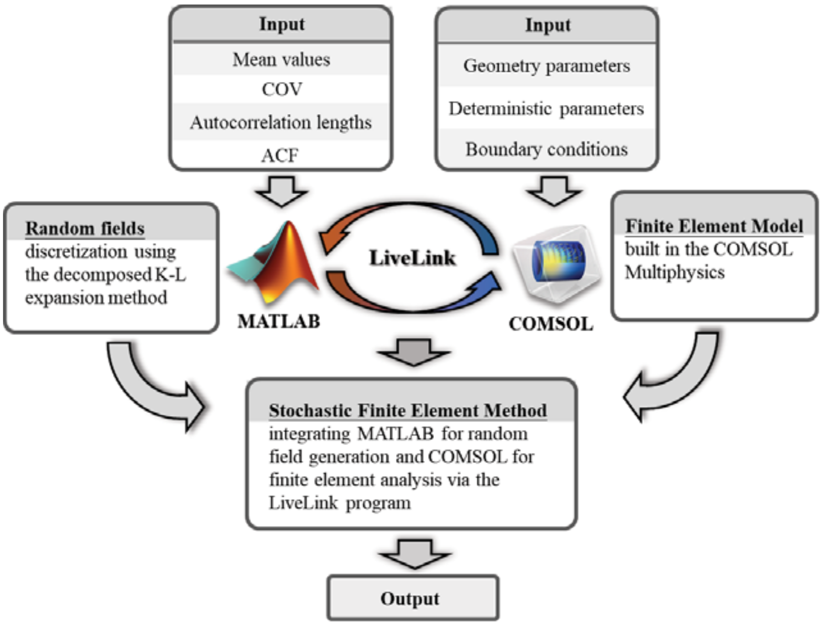

Figure 1. The framework of SFEM integrating MATLAB and COMSOL via the LiveLink program.

The proposed decomposed K-L expansion method could then be applied in the SFEM for geotechnical problems to take the spatial variability of soil properties into consideration (Gong et al., 2020; Li et al., 2020; Pan et al., 2021; Zhang et al., 2021). The SFEM which combined the finite element analysis in COMSOL and the random field discretization in MATLAB was developed by coding a LiveLink program (Zhu et al., 2021). A schematic of the framework is shown in Figure 1. Firstly, the parameters of the statistical features and the correlation structures used for the generation of random fields were imported into MATLAB for the procedure described in section 2.1. The geometry parameters, deterministic parameters, and the boundary conditions that were used to define the geotechnical model were input into COMSOL for finite element analysis. The generated random field was imported into COMSOL by the LiveLink program for the consideration of the heterogeneity of the soil property. Finally, the results of SFEM were obtained, which can be input into the MATLAB for further stochastic analysis.

3 Verification of the proposed decomposed method

3.1 Verification of the accuracy

To validate the accuracy of the decomposed K-L expansion scheme for 3-D random field discretization, an illustrative example was carried out. In the example, the object property was set to be log-normally distributed in space and the ACF was set as the single exponential function with correlation lengths being $l_x = l_y = 10$ m, $l_z = 5$ m. The proposed decomposed K-L expansion scheme was employed for the 3-D random field generation. One typical realization of the random field is shown in Figure 2. To verify the statistical properties and the correlation structures of the simulated random field, the statistical pattern and the correlations along vertical and horizontal directions are estimated by the statistical analysis, the results of which are compared with the corresponding true ones. The histogram showing the parameter distribution of the generated 3-D random field from statistical sampling and the plot showing the actual parameter distribution used for the random field definition are compared and shown in Figure 2. It's clear that the generated random field represents the statistical property of the parameter in space very well. The simulated ACF of the random field realization based on spatial averaging was compared to the analytical expression of ACF. The simulated ACF and analytical ACF that describes the parameter correlation along vertical and horizontal directions are shown in Figure 3. It is clear that the ACFs plotted as the function of space gap, which describes the correlation structures of the simulated random field, are in good agreement with the analytical results.

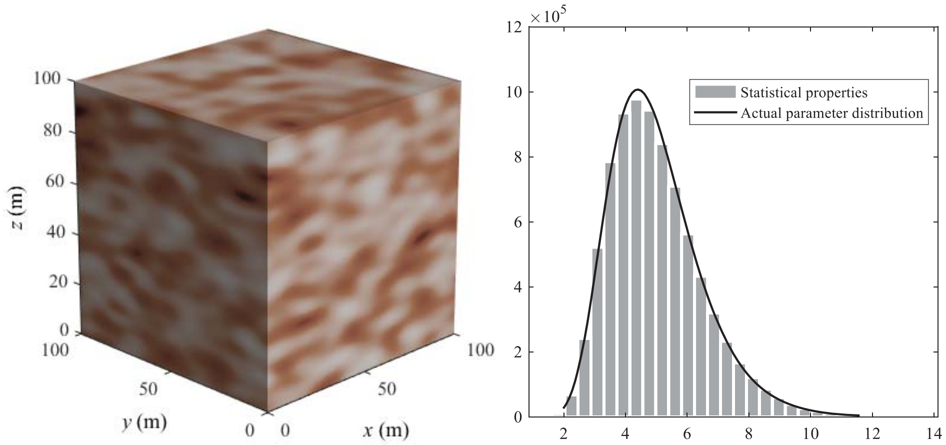

Figure 2. One typical generation of the 3-D random field in the illustrative case and the validation of the parameter statistical distributions.

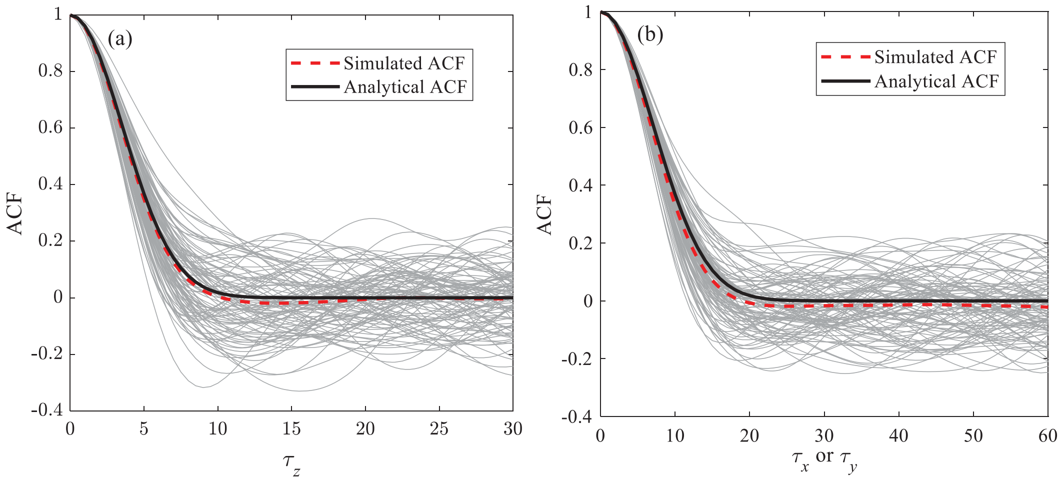

Figure 3. The validation of the autocorrelation structures along vertical (a) and horizontal (b) directions.

3.2 Verification of the efficiency

Contrastive analyses of the computing time and the storage space in use are conducted between the proposed method and the conventional K-L expansion to further validate the computational efficiency of the decomposed K-L expansion. In the illustrative examples, the single exponential ACF is applied in the random field discretization with the correlation lengths in each direction set as $l_x = 10$ m, $l_y = 10$ m, $l_z = 5$ m. The geometric size of the 3-D random fields varies from (5 m × 5 m × 5 m) to (600 m × 600 m × 600 m), and the mesh spacing in all three directions are kept at 1 m. As a result, the number of grid points varies from 125 to 2.16×10^8. The numerical solutions of eigenpairs are employed in the K-L expansion, and an adequate number of truncated terms is used to guarantee the accuracy of the series expansion. The computing time and storage space required for the discretization of random fields with different numbers of nodes on a desktop computer with a 3.60 GHz i9-10850K Core CPU and 32 GB of RAM are shown in Figure 4. When the traditional K-L method is used to discretize a random field with the number of grid points greater than (25 × 25 × 25), the memory usage will exceed the configured threshold, and the program is unable to produce the expected random field. As the number of nodes increases to a certain extent, the computation time and memory usage in random field generation are mainly contributed by the matrix decomposition rather than other functions, therefore, in the double logarithm coordinate as shown in Figure 4, when the number of nodes is greater than about 10^6, the computation time and memory usage greatly increase with the number of nodes. As can be seen from these figures, the proposed decomposed K-L expansion scheme significantly reduces the computing time and physical memory requirements, which makes the discretization of a 3-D random field with high definition and large geometric size possible.

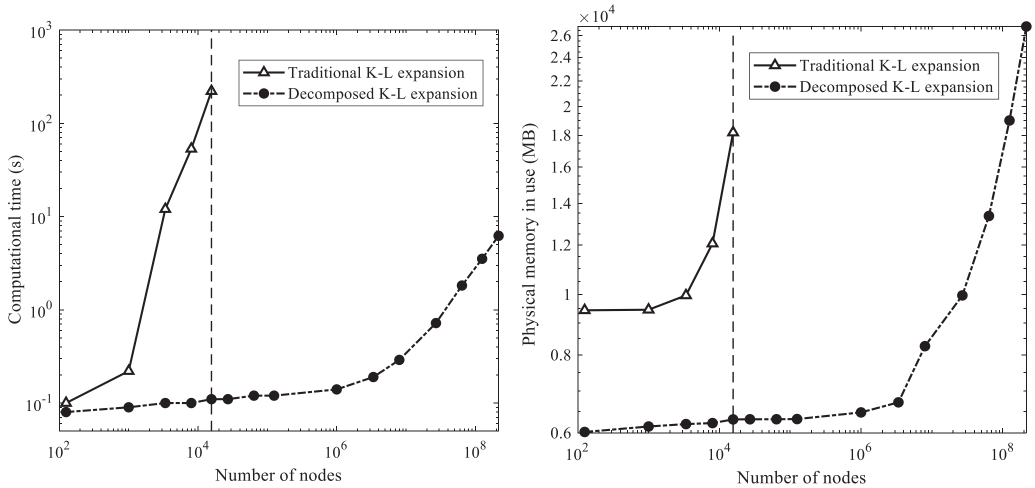

Figure 4. Comparisons of the computing time and storage space between the decomposed K-L expansion method and the traditional method.

4 Conclusion

In the present study, a decomposed K-L expansion method was proposed for the discretization of 3-D random fields with separable ACFs. In the proposed scheme, the generation of a 3-D random field will be decomposed into that of separate 1-D random fields, and the eigen solutions needed for the random field discretization could be solved using the autocorrelation matrix in separate directions, making it much more efficient for the discretization of 3-D random fields. The accuracy of the discretized random field and the efficiency of the proposed scheme were demonstrated. Compared with the traditional method, the proposed decomposed K-L expansion scheme significantly reduces the computing time and storage space, which makes the discretization of 3-D random fields more efficient and the simulation of multidimensional random fields with high definition and large geometric size into possible.

More comparisons between the proposed method and traditional method and the application of the decomposed K-L expansion to the stochastic finite element analysis of geotechnical problems will be carried out in our future studies.

Acknowledgments

This work was supported by the Fundamental Research Funds for the Central Universities [No. 2022QN1026] and the National Natural Science Foundation of China [No. 52122805, No. 51890912].

References

Betz, W., Papaioannou, I., and Straub, D. (2014). Numerical methods for the discretization of random fields by means of the Karhunen–Loève expansion. *Computer Methods in Applied Mechanics and Engineering*, 271, 109-129.

Cheng, H., Chen, J., Chen, R., Huang, J., and Li, J. (2019). Three-dimensional analysis of tunnel face stability in spatially variable soils. *Computers and Geotechnics*, 111, 76-88.

Gong, W., Tang, H., Juang, C. H., and Wang, L. (2020). Optimization design of stabilizing piles in slopes considering spatial variability. *Acta Geotechnica*, 15(11), 3243-3259.

Griffiths, D. V., Huang, J., and Fenton, G. A. (2009). Influence of Spatial Variability on Slope Reliability Using 2-D Random Fields. *Journal of Geotechnical and Geoenvironmental Engineering*, 135(10), 1367-1378.

Jiang, S.-H., Huang, J., Griffiths, D. V., and Deng, Z.-P. (2022). Advances in reliability and risk analyses of slopes in spatially variable soils: A state-of-the-art review. *Computers and Geotechnics*, 141, 104498.

Li, C. F., Feng, Y. T., Owen, D. R. J., Li, D. F., and Davis, I. M. (2008). A Fourier-Karhunen-Loève discretization scheme for stationary random material properties in SFEM. *International Journal for Numerical Methods in Engineering*, 73(13), 1942-1965.

Li, D.-Q., Xiao, T., Zhang, L.-M., and Cao, Z.-J. (2019). Stepwise covariance matrix decomposition for efficient simulation of multivariate large-scale three-dimensional random fields. *Applied Mathematical Modelling*, 68, 169-181.

Li, D. Q., Wang, M. X., and Du, W. Q. (2020). Influence of spatial variability of soil strength parameters on probabilistic seismic slope displacement hazard analysis. *Engineering Geology*, 276, 105744.

Liu, Y., Lee, F.-H., Quek, S.-T., and Beer, M. (2014). Modified linear estimation method for generating multi-dimensional multi-variate Gaussian field in modelling material properties. *Probabilistic Engineering Mechanics*, 38, 42-53.

Pan, Y., Liu, Y., Tyagi, A., Lee, F.-H., and Li, D.-Q. (2021). Model-independent strength-reduction factor for effect of spatial variability on tunnel with improved soil surrounds. *Géotechnique*, 71(5), 406-422.

Phoon, K.-K. and Kulhawy, F. H. (1999a). Characterization of geotechnical variability. *Canadian Geotechnical Journal*, 36(4), 612-624.

Phoon, K.-K. and Kulhawy, F. H. (1999b). Evaluation of geotechnical property variability. *Canadian Geotechnical Journal*, 36(4), 625-639.

Sudret, B. and Der Kiureghian, A. (2002). Comparison of finite element reliability methods. *Probabilistic Engineering Mechanics*, 17(4), 337-348.

Tsantili, I. C. and Hristopulos, D. T. (2016). Karhunen–Loève expansion of Spartan spatial random fields. *Probabilistic Engineering Mechanics*, 43, 132-147.

Xiao, T., Li, D.-Q., Cao, Z.-J., Au, S.-K., and Phoon, K.-K. (2016). Three-dimensional slope reliability and risk assessment using auxiliary random finite element method. *Computers and Geotechnics*, 79, 146-158.

Yue, Q., Yao, J., Ang, A. H. S., and Spanos, P. D. (2018). Efficient random field modeling of soil deposits properties. *Soil Dynamics and Earthquake Engineering*, 108, 1-12.

Zhang, L. L., Cheng, Y., Li, J. H., Zhou, X. L., Jeng, D. S., and Peng, X. Y. (2016). Wave-induced oscillatory response in a randomly heterogeneous porous seabed. *Ocean Engineering*, 111, 116-127.

Zhang, W.-g., Meng, F.-s., Chen, F.-y., and Liu, H.-l. (2021). Effects of spatial variability of weak layer and seismic randomness on rock slope stability and reliability analysis. *Soil Dynamics and Earthquake Engineering*, 146, 106735.

Zheng, Z. and Dai, H. (2017). Simulation of multi-dimensional random fields by Karhunen–Loève expansion. *Computer Methods in Applied Mechanics and Engineering*, 324, 221-247.

Zhu, B., Hiraishi, T., Mase, H., Baba, Y., Pei, H., and Yang, Q. (2021). A 3-D numerical study of the random wave-induced response in a spatially heterogenous seabed. *Computers and Geotechnics*, 135, 104159.