Proc. of the 8th International Symposium on Geotechnical Safety and Risk (ISGSR)
Edited by Jinsong Huang, D.V. Griffiths, Shui-Hua Jiang, Anna Giacomini and Richard Kelly
 Published by Research Publishing, Singapore.
doi: 10.3850/978-981-18-5182-7_06-012-cd

ISGSR 2022

An Efficient Adaptive Response Surface Method for Reliability Analysis of Geotechnical Engineering Systems Using Adaptive Bayesian Compressive Sensing and Monte Carlo Simulation (ABCS-MCS)

Peiping Li[1*] and Yu Wang[2]

[1*]Department of Architecture and Civil Engineering, City University of Hong Kong, Tat Chee Avenue, Hong Kong.
E-mail: peiping.li@my.cityu.edu.hk (Corresponding author)
[2]Department of Architecture and Civil Engineering, City University of Hong Kong, Tat Chee Avenue, Hong Kong.
E-mail: yuwang@my.cityu.edu.hk

Abstract: Geotechnical reliability-based design method provides a systematic framework to consider the effect of input parameter uncertainty on geotechnical structure design. A main challenge of reliability-based design for geotechnical engineering systems is the computational power required for reliability assessment when time-consuming geotechnical deterministic models are used, for example, finite element model. Direct use of Monte Carlo simulation (MCS) would lead to significant computational cost, which is inefficient and even unrealistic. Response surface methods (RSMs) have been developed to improve the efficiency of reliability analysis. However, most RSMs cannot self-evaluate the accuracy of reliability analysis results. Therefore, this renders a challenging question in RSMs applications that how to determine whether the number of sampling points is sufficient to achieve a target accuracy of reliability analysis. Using the advantage of self-estimated uncertainty of Kriging method, adaptive Kriging MCS (AK-MCS) method adopts the Kriging model to construct a response surface and combines a learning function to sequentially select additional sampling points to improve the accuracy of reliability analysis until a target accuracy is achieved. However, AK-MCS requires extensive sampling data to construct the reliable trend function and auto-correlation structure function, and it is also not applicable to highly non-stationary data. To address these challenges, an innovative adaptive response surface method is developed using adaptive Bayesian compressive sensing (ABCS) and Monte Carlo simulation (MCS), called ABCS-MCS. ABCS-MCS can self-evaluate the uncertainty of predictions and combine a learning function to adaptively determine the minimum number of sampling points and their locations for achieving a target accuracy of reliability analysis. ABCS-MCS is directly applicable to non-stationary data because it is a purely non-parametric method. A two-layered slope reliability analysis problem with consideration of spatial variability in soil properties is illustrated. Results demonstrate that ABCS-MCS outperforms AK-MCS in terms of accuracy and efficiency of reliability analysis.

Keywords: Reliability analysis; small failure probability; adaptive response surface method; Bayesian compressive sensing; non-parametric method.

1 Introduction

Geotechnical reliability-based design method provides a systematic framework to consider the effect of input parameter uncertainty on geotechnical structure design. A main challenge of reliability-based design for geotechnical engineering systems is the computational power required for reliability assessment when complex geotechnical deterministic models are used, for example, finite element model (FEM). For the deterministic model with time-consuming computation of response, direct use of Monte Carlo simulation (MCS) would lead to significant computational cost, which is inefficient and even unrealistic. In addition, complex geotechnical systems usually have high-dimensional input parameters and highly nonlinear output response behavior (e.g., multiple-layered soil slope system), commonly used reliability analysis methods, such as First Order Reliability method and Second Order Reliability method, are unsuitable for these high-dimensional or highly nonlinear problems (e.g., Huang et al. 2016). Some variance reduction methods have been developed to improve the efficiency of reliability analysis, such as Importance Sampling and Subset Simulation. However, it still requires a rather large number of sampling points to directly use these variance reduction methods. To further improve the efficiency of reliability analysis, response surface methods (RSMs) have been developed to build a surrogate model to approximate a complicated model, leading to an easy-to-evaluate mathematical model for performing MCS. However, most RSMs cannot self-evaluate the accuracy of reliability analysis results and rely on independent MCS runs on the original systems for verification. This hinders RSM applications and proposes a challenging question in RSM applications that how to determine whether or not the number of sampling points is sufficient to achieve a target accuracy of reliability analysis.

Leveraging on the advantage of being able to self-estimate uncertainty of Kriging method, many adaptive Kriging (AK) RSMs have been developed in literature for address this problem (e.g., Afshari et al. 2021;

Teixeira et al. 2021). One example of AK RSMs is adaptive Kriging MCS (AK-MCS) (e.g., Echard et al. 2011). It uses the Kriging model to construct a response surface and combines a learning function to sequentially select additional sampling points to improve the accuracy of reliability analysis until a target accuracy is achieved. However, Kriging method generally requires extensive sampling data to construct a reliable trend function and auto-correlation function. Besides, stationary assumptions should be made when using Kriging method, therefore, AK-MCS is not directly applicable for highly non-stationary data.

To address the challenges mentioned above, an innovative adaptive RSM based on adaptive Bayesian Compressive sensing (ABCS) and MCS is presented in this study, denoted as ABCS-MCS (Li and Wang 2022). BCS is a novel sampling strategy which can reconstruct the complete information of the signal from sparse measurements of the signal (e.g., Wang and Zhao 2017). Besides, BCS does not incorporate any trend function or auto-correlation structure function because it is a purely non-parametric method (e.g., Wang et al. 2018&2019; Zhao and Wang 2020), hence it is directly to the non-stationary data. Furthermore, BCS can provide the high-accuracy response predictions at any unsampled locations and quantify explicitly the associated prediction uncertainty. Thus, BCS is combined with a learning function to develop an adaptive BCS (ABCS) RSM for adaptively determine the minimum number of sampling points and their locations for achieving a target accuracy of reliability analysis. After this introduction, framework of the proposed ABCS-MCS method is first described. Then, equations for the BCS RSM are derived, followed by a brief review of the learning function used in the proposed method, i.e., U-learning function (e.g., Echard et al. 2011). Finally, a comparative study between the proposed method and AK-MCS is performed using a two-layered slope reliability analysis problem with consideration of spatial variability in soil properties.

2 Framework of the proposed method for reliability analysis

The objective of the proposed ABCS-MCS method is to adaptively determine the minimum number of sampling points and their sampling locations for achieving a target accuracy of reliability analysis. The basic idea of ABCS-MCS is to first generate a random sample set $\boldsymbol{S}$ without calculating their corresponding response. Then, BCS method is used to train a response surface using sparse sampling points, to provide the response predictions of $\boldsymbol{S}$ and uncertainties of these predictions. Subsequently, a learning function is combined with the BCS response surface results to adaptively identify the best additional sampling points in $\boldsymbol{S}$ for re-training the BCS response surface, improving the accuracy of reliability analysis. This process repeats until the target accuracy of reliability analysis is achieved.

Figure 1 shows the framework of the proposed ABCS-MCS method for reliability analysis, which involves seven main steps. In Step 1, N_{MCS} sets of random samples are generated based on the probability distributions of input random variables without calculating their responses using the original model, and the generated samples are denoted as sample set $\boldsymbol{S}$. In Step 2, Latin Hypercube Sampling (LHS) method (e.g., McKay et al. 1979) is adopted to generate M sampling points as the initial experimental design (ED), and the responses of these M sampling points are estimated using the original model. In Step 3, a BCS-based response surface is trained using the ED for approximating the original model. In Step 4, the constructed BCS response surface is used to estimate response predictions and their corresponding uncertainties for all samples in $\boldsymbol{S}$, which are subsequently used to estimate the corresponding failure probability. In Step 5, the BCS response surface results from Step 4 are combined with a learning function to calculate the learning function values for all the samples in $\boldsymbol{S}$. The best additional sample $\boldsymbol{X}^*$ in $\boldsymbol{S}$ is identified via a learning criterion, which depends on the learning function chosen as discussed in section 4. In Step 6, the learning function value of $\boldsymbol{X}^*$ is compared with a stopping condition for active learning. When the stopping condition in Step 6 is unsatisfied, the response at $\boldsymbol{X}^*$ is evaluated using the original model and ED is updated by adding the $\boldsymbol{X}^*$, and Steps 3-6 are then repeated until the stopping condition is satisfied. When stopping condition is satisfied, the trained BCS response surface is considered accurate enough for providing the response predictions of all samples in $\boldsymbol{S}$. Step 7 checks whether $\boldsymbol{S}$ is sufficiently large to obtain an acceptable coefficient of variation (COV) on the failure probability (P_f) obtained from the constructed BCS response surface. The COV is calculated as (e.g., Ang and Tang 2007):

$$\mathrm{COV}_{P_f} = \sqrt{\frac{1-P_f}{P_f N_{MCS}}} \tag{1}$$

In this study, a COV below 5% is seen as acceptable. When the calculated COV exceeds 5%, $\boldsymbol{S}$ is enriched by increasing additional N_{MCS} sets samples. Then, the P_f is re-estimated using the updated $\boldsymbol{S}$, and Steps 4-7 are repeated until the COV on the failure probability is less than the target threshold. Finally, the last estimation of failure probability is used as the reliability analysis result of the proposed method.

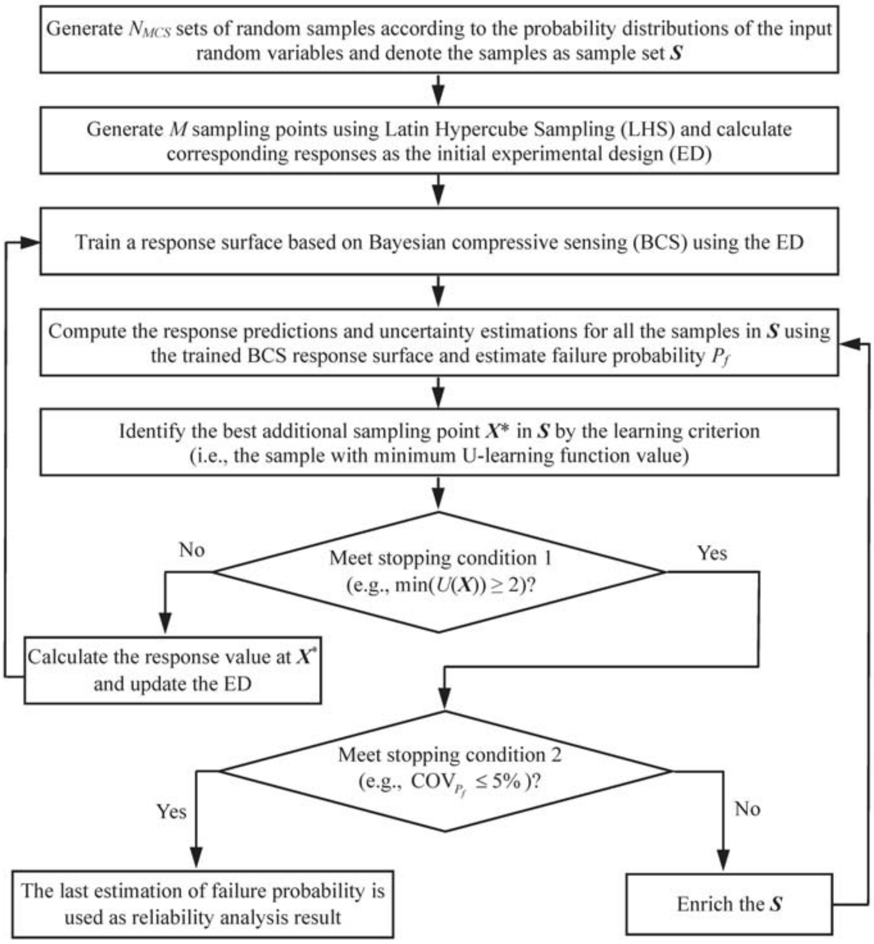

Figure 1. Framework of the proposed method for reliability analysis (after Li and Wang 2022).

3 Bayesian compressive sensing-based response surface method

BCS is a novel sampling theory to reconstruct the complete information of one signal from its sparse sampling data. It can provide both the response predictions at unsampled points and their associated uncertainties. According to the context of compressive sensing, the n-dimensional (nD) dataset of the system responses (i.e., $\underset{\sim}{\mathbf{F}}$ with dimensions of $N_1 \times N_2 \times \ldots \times N_n$) for a system with n input random variables, i.e., $\boldsymbol{X} = (X^1, X^2, \ldots, X^n)$, can be represented as a weighted summation of many nD basis functions (e.g., Li and Wang 2022). Mathematically, $\underset{\sim}{\mathbf{F}} = \sum_{t=1}^{N} \underset{\sim}{\mathbf{B}}_t^{n\mathrm{D}} \omega_t^{n\mathrm{D}}$, where $N = N_1 \times N_2 \times \ldots \times N_n$, $\underset{\sim}{\mathbf{B}}_t^{n\mathrm{D}}$ is the t-th nD basis function and $\omega_t^{n\mathrm{D}}$ is the weight corresponding to $\underset{\sim}{\mathbf{B}}_t^{n\mathrm{D}}$. Note that $\underset{\sim}{\mathbf{B}}_t^{n\mathrm{D}}$ is independent of $\underset{\sim}{\mathbf{F}}$ and has the same dimension as $\underset{\sim}{\mathbf{F}}$. $\underset{\sim}{\mathbf{B}}_t^{n\mathrm{D}}$ can be constructed from the columns of n independent one-dimensional (1D) discrete cosine basis function matrices, $\mathbf{B}^1$, $\mathbf{B}^2$, …, and $\mathbf{B}^n$, corresponding to $X^1, X^2, \ldots, X^n$, respectively (e.g., Li and Wang 2021). The dimensions of $\mathbf{B}^1$, $\mathbf{B}^2$, …, and $\mathbf{B}^n$ are $N_1 \times N_1$, $N_2 \times N_2$, …, and $N_n \times N_n$, respectively. For a compressible signal, most elements in $\boldsymbol{\omega}^{n\mathrm{D}} = (\omega_1^{n\mathrm{D}}, \omega_2^{n\mathrm{D}}, \ldots, \omega_N^{n\mathrm{D}})$ are almost zero except a few nontrivial ones. Therefore, the nontrivial coefficients in $\boldsymbol{\omega}^{n\mathrm{D}}$ can be estimated using M sparse sampling data $\boldsymbol{y}^{n\mathrm{D}}$ ($M < N$). Once the nontrivial coefficients in $\boldsymbol{\omega}^{n\mathrm{D}}$ are estimated using $\boldsymbol{y}^{n\mathrm{D}}$, $\boldsymbol{\omega}^{n\mathrm{D}}$ can be approximated by setting the trivial elements of $\boldsymbol{\omega}^{n\mathrm{D}}$ to zero, i.e, $\omega_t^{n\mathrm{D}}$ ($t = 1, 2, \ldots, N$) is approximated as $\hat{\omega}_t^{n\mathrm{D}}$. In such cases, the system response dataset $\underset{\sim}{\mathbf{F}}$ can be reconstructed as $\underset{\sim}{\hat{\mathbf{F}}}$, and it is expressed as:

$$\underset{\sim}{\hat{\mathbf{F}}} = \sum_{t=1}^{N} \underset{\sim}{\mathbf{B}}_t^{n\mathrm{D}} \hat{\omega}_t^{n\mathrm{D}} \tag{2}$$

Li and Wang (2022) shows that $\hat{\omega}_t^{n\mathrm{D}}$ ($t = 1, 2, \ldots, N$) can be estimated under a Bayesian framework based on the Markov Chain Monte Carlo (MCMC) simulation approach using $\boldsymbol{y}^{n\mathrm{D}}$, and $\hat{\omega}_t^{n\mathrm{D}}$ follows a multivariate

Gaussian distribution. The mean and standard deviation (SD) of system response dataset can be obtained using the generated MCMC samples of $\hat{\omega}_t^{nD}$ based on Eq. (2).

4 Learning function

In this study, U-learning function (e.g., Echard et al. 2011) is adopted in the proposed ABCS-MCS method to adaptively identify the best additional sampling point in $\boldsymbol{S}$ to improve the accuracy of reliability analysis. As mentioned above, the estimated $\hat{\omega}_t^{nD}$ (t = 1, 2, …, N) follows a multivariate Gaussian distribution, and $\underset{\sim}{\mathbf{B}}_t^{nD}$ is independent of $\hat{\omega}_t^{nD}$, and therefore, the response surface predictions for $\boldsymbol{S}$ also follow a multivariate Gaussian distribution. According to the mean (i.e., $\mu_{\hat{\underset{\sim}{F}}}(\boldsymbol{X})$) and corresponding SD (i.e., $\sigma_{\hat{\underset{\sim}{F}}}(\boldsymbol{X})$) of system responses obtained from BCS-based RSM, U-learning function indicates a distance in terms of SD between the mean and limit state value (i.e., F_{LS}), which is expressed as:

$$U(\boldsymbol{X}) = \frac{\left| \mu_{\hat{\underset{\sim}{F}}}(\boldsymbol{X}) - F_{LS} \right|}{\sigma_{\hat{\underset{\sim}{F}}}(\boldsymbol{X})} \tag{3}$$

$U(\boldsymbol{X})$ can be regarded as an equivalent reliability index which reflects the probability of the predicted response of a sample being misclassified into the wrong performance domain (e.g., safe samples being classified to a failure domain, or vice versa). A small value of $U(\boldsymbol{X})$ means that predicted response from BCS-based RSM has a high probability of being misclassified to the incorrect performance domain. Therefore, the learning criterion is defined as to select a point with the minimum value of $U(\boldsymbol{X})$ in $\boldsymbol{S}$ as the best additional sampling point $\boldsymbol{X}^*$. Besides, the stopping condition of adaptive sampling is defined as when the minimum $U(\boldsymbol{X})$ value exceeds a threshold value e.g., min($U(\boldsymbol{X})$) ≥ 2. According to the physical meaning of the reliability index, $U(\boldsymbol{X})$ = 2 corresponds to a probability of Φ(2) = 97.7% that the sample point $\boldsymbol{X}$ has been classified into the correct performance domain.

5 Application example

The proposed ABCS-MCS method is applied to reliability analysis of a two-layered cohesive slope with consideration of spatial variability in soil property. Figure 2 shows the geometry of the slope, which has been studied by Li and Wang (2021). As summarized in Table 1, the soil unit weight γ of slope is considered as a constant value of γ = 19 kN/m^3. The undrained shear strength of two clay layers, S_{u1} and S_{u2}, are modeled by two 2D lognormal stationary random fields, respectively. The mean and coefficient of variation (COV) of S_{u1} are 36 kPa and 0.3, respectively, while the mean and COV of random field S_{u2} are 55 kPa and 0.3, respectively. The auto-correction structure of the two random fields for S_{u1} and S_{u2}, were both modeled by a squared exponential autocorrection function, and they are independent. The horizontal and vertical scale of fluctuations used for simulating the random fields of S_{u1} and S_{u2} are 50m and 5m, respectively. The two independent lognormal stationary random fields of S_{u1} and S_{u2} are simulated using Karhunen-Loève (KL) expansion. In this study, when a squared exponential autocorrelation function with δ_h = 50m and δ_v = 5m are used to simulate the random field of S_{u1} and S_{u2}, the truncated number of KL expansion terms are 4 and 6 for S_{u1} and S_{u2}, respectively. Therefore, there are 10 independent standard normal random variables as input random variables in the example.

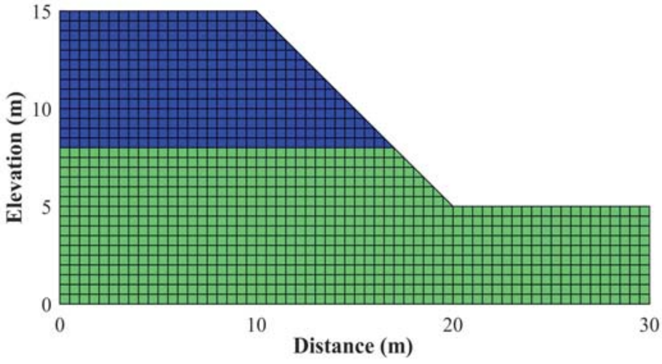

Figure 2. Geometry and finite element mesh of the illustrative cohesive slope.

Table 1. Statistical properties of soil parameters for the illustrative two-layered cohesive slope.

Parameter	Mean	COV*	Distribution
Undrained shear strength S_{u1}	36 kPa	0.3	Lognormal
Undrained shear strength S_{u2}	55 kPa	0.3	Lognormal
Soil unit weight γ	19 kN/m^3	-	-

Note: *COV represents coefficient of variation

To use the proposed ABCS-MCS method for reliability analysis of the illustrative slope, a sample set $\boldsymbol{S}$ of N_{MCS} = 10,000 random samples is first generated. The samples of these 10 standard normal random variables are generated from the range of [-4, 4]. Then, M = 150 sampling points are generated using the LHS technique as the initial experiment design to construct a BCS-based RSM. The response surface resolutions along each random variable are all defined as $N_1 = N_2 = \ldots = N_{10} = 64$. Furthermore, 1,000 independent MCMC simulation samples of $\hat{\omega}_t^{nD}$ (t = 1, 2, …, N) are used to calculate the mean and corresponding standard deviation (SD). The slope failure probability obtained from MCS using $\boldsymbol{S}$ with the finite element model solved by strength reduction method is P_{f_MCS} = 0.0709, which is considered as the exact solution for comparison. In addition, reliability analysis result of slope obtained from the AK-MCS method is also provided for comparison.

Figure 3 shows the results of adaptive reliability analysis process for the slope example using ABCS-MCS and AK-MCS. ABCS-MCS requires 998 sampling points to satisfy stopping condition, and the obtained failure probability is 0.0711, which is very close the P_{f_MCS} (see the black dashed line). While AK-MCS requires 1,186 sampling points to reach convergence, and the failure probability obtained from AK-MCS is 0.0699. This indicates that ABCS-MCS is more efficient and accurate than AK-MCS for the slope reliability analysis.

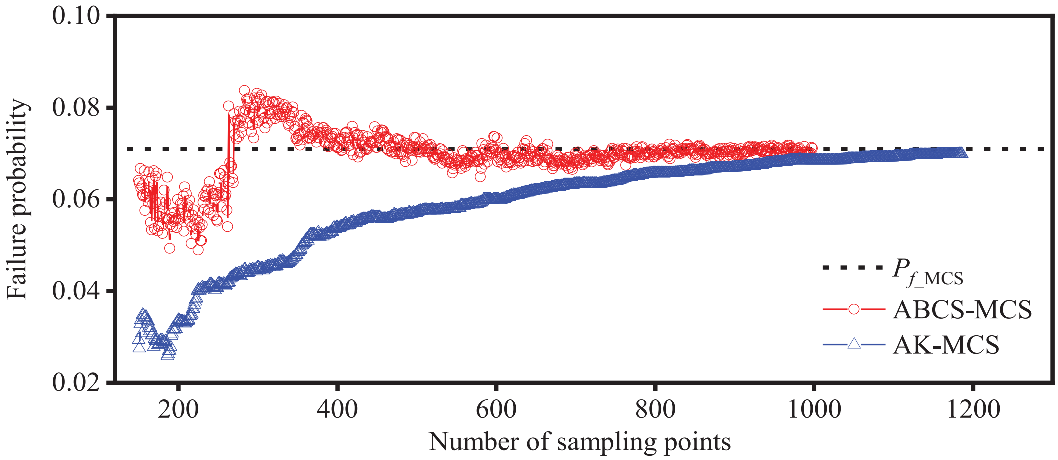

Figure 3. Comparison of adaptive reliability analysis process using ABCS-MCS and AK-MCS for the slope example.

6 Conclusion

An innovative adaptive response surface method, based on adaptive BCS and MCS and called ABCS-MCS, was presented in this paper. The proposed ABCS-MCS method can provide both response predictions at unsampled points and quantify explicitly the associated prediction uncertainty, and it is combined with the U-learning function to adaptively determined the minimum number of sampling points and corresponding sampling locations for achieving a target accuracy of reliability analysis. The proposed ABCS-MCS method is illustrated using a two-layered cohesive slope with consideration of spatially varying soil properties. Results shows that the proposed ABCS-MCS is more accurate and efficient than AK-MCS to perform the reliability analysis of slope stability.

Acknowledgments

The work described in this paper was supported by grants from the Research Grant Council of Hong Kong Special Administrative Region, China (Project nos. CityU 11213119 and C6006-20G). The financial support is gratefully acknowledged.

References

Afshari, S., Enayatollahi, F., Xu, X., and Liang, X. (2021) Machine learning-based methods in structural reliability analysis: A Review. *Reliability Engineering and System Safety*, 206: 107285.

Ang, A.H.-S., and Tang, WH. (2007) Probability Concepts in Engineering: Emphasis on Applications to Civil and Environmental Engineering. *John Wiley & Sons, New York*.

Echard, B., Gayton, N., and Lemaire, M. (2011) AK-MCS: An active learning reliability method combining Kriging and Monte Carlo Simulation. *Structural safety*, 33: 145-154.

Huang, X., Chen, J., and Zhu, H. (2016) Assessing small failure probabilities by AK-SS: An active learning method combining Kriging and Subset Simulation. *Structural Safety*, 59: 86-95.

Li, P. and Wang, Y. (2021) Development of an efficient response surface method for highly nonlinear systems from sparse sampling data using Bayesian Compressive Sensing, *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering,* 7 (4): 04021050.

Li, P. and Wang, Y. (2022) An active learning reliability analysis method using adaptive Bayesian compressive sensing and Monte Carlo Simulation (ABCS-MCS). *Reliability Engineering and System Safety*, 59: 108377.

McKay, M.D., Conover, W.J., and Beckman, R.J. (1979) A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics*, 21 (2): 239-245.

Teixeira, R., Nogal, M., and O'Connor, A. (2021) Adaptive approaches in metamodel-based reliability analysis: A review. *Structural Safety*, 89: 102019.

Wang, Y. and Zhao, T. (2017) Statistical interpretation of soil property profiles from sparse data using Bayesian compressive sampling. *Géotechnique*, 67 (6): 523-536.

Wang, Y. Zhao, T., and Phoon, K.K. (2018) Direct simulation of random field samples from sparsely measured geotechnical data with consideration of uncertainty in interpretation. *Canadian Geotechnical Journal*, 55 (6): 862-880.

Wang, Y. Zhao, T., Hu, Y., and Phoon, K.K., (2019) Simulation of random fields with trend from sparse measurements without detrending. *Journal of Engineering Mechanics*, 145 (2): 04018130.

Zhao, T. and Wang, Y. (2020) Non-parametric simulation of non-stationary non-Gaussian 3D random field samples directly from sparse measurements using signal decomposition and Markov Chain Monte Carlo. *Reliability Engineering and System Safety*, 203: 107087.