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Regional Reliability Sensitivity Analysis Considering Spatial Variability of Soil

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Abstract: The influence of spatial variability of soil on geotechnical engineering such as the ultimate bearing capacity of shallow foundation can be considered by reliability analysis based on random field theory. However, reliability analysis can only estimate the reliability index or failure probability of geo-structures, which cannot identify the important regions of random fields. Hence, a reliability sensitivity analysis method considering spatial variability of soil (called RSA–KL–FORM–*x*) is proposed in this study, which employs the Karhunen–Loève (KL) expansion method and the first–order reliability method (FORM) as the method of realizing random field and reliability analysis method, respectively. In the proposed RSA–KL–FORM–*x*, mean reliability sensitivity index (MRSI) and standard deviation reliability sensitivity index (SDRSI) are defined as functions of coordinates in random field domain, which means that the MRSI (or the SDRSI) with larger absolute values can be regarded as important regions. A shallow foundation with spatially various cohesion of soil is adopted to illustrate the process of the RSA–KL–FORM–*x*. The results show that the top part of the soil in contact with the shallow foundation is an important region to improve the reliability of the shallow foundation.

Keywords: Reliability analysis; Reliability sensitivity analysis; Random field; Shallow foundation.

1 Introduction

During the last several decades, effects of random uncertainties on geotechnical engineering have been investigated. Therein, reliability analysis aims at assessing the performance of a geo-structure model by calculating its reliability index β, while reliability sensitivity analysis aims at investigating the effects of random variables on the reliability index β of a geo-structure model (Li et al., 2009). Generally, reliability sensitivity analysis is mostly based on corresponding reliability analysis, which can be divided into simulation–based methods and approximately analytical methods (Huang and Zhang, 2013). Among many methods, the analytical solution of reliability sensitivity analysis based on the first–order reliability method (FORM) is often employed due to satisfactory computational efficiency and reasonable computational accuracy (Ditlevsen and Madsen, 1996). Because a random field is composed of infinite correlated random variables, these methods adapted in random variable model had hardly been extended to random field model that is of interest to researchers and engineers.

To overcome this difficulty, a reliability sensitivity analysis method considering spatial variability of soil (called RSA–KL–FORM–*x*) is proposed to identify the important regions of random field, where the Karhunen–Loève (KL) expansion method is employed as a method of realizing random field and the FORM is employed as a reliability analysis method. After the theory of reliability analysis and reliability sensitivity analysis are explained, an example of shallow foundation is employed to illustrate the process and the merits of the RSA–KL–FORM–*x*.

2 Reliability analysis based on the KL expansion method

A random field is composed of infinite correlated random variables, where the value of a variable at one point is correlated to the values at nearby points. To carry out reliability analysis considering random field, a random field must be represented by a finite number of random variables, which is call random field discretization. The KL expansion method one of series expansion methods can realize random fields with fewer random variables than point discretization methods and average discretization methods (Sudret and Der Kiureghian, 2000). Meanwhile, to reduce the number of calculations of limit state function, the FORM is a better choice than the theoretically more accurate Monte Carlo simulation (Wang, 2011). Hence, combination of the KL expansion method and the FORM (i.e., the KL–FORM) is proposed in Fei et al. (2021). Note that the standard normal random variable vector $\boldsymbol{\xi}$ in the KL expansion method is considered as the random variable vector for reliability analysis. However, the convergence of the FORM has long plagued designers and engineers. For guaranteeing excellent numerical stability of the FORM, the improved Hasofer–Lind–Rackwitz–Fiessler–*x* (iHLRF–*x*) (Ji et al., 2018) is introduced into the iterative process of the FORM (called KL–FORM–*x*).

2.1 KL expansion method

In geotechnical engineering, spatial variability of soil can be described by various types of auto-correlation functions (ACFs). In this study, a one-dimensional single exponential (SNX) ACF along the vertical direction is used to express the auto-correlation between any two locations in a random field domain Ω:

$$\rho(y, y') = \exp(-(|y - y'|)/\theta) \tag{1}$$

where θ is auto-correlation distance (ACD). After the ACF has been determined, the KL expansion method realizes the random field by the spectral decomposition of the ACF, whose procedures for a one-dimensional random field are summarized as follows.

1) Calculate the eigenvalues λ_i and eigenfunctions $\phi_i(y)$ ($i = 1, 2, \ldots, M$; and M is the number of KL expansion terms) of the random field by solving the homogeneous Fredholm integral equation according to the ACF. However, λ_i and $\phi_i(y)$ cannot be calculated analytically in most cases, so the Jacobi–Lagrange–Galerkin (JLG) method, which is a numerical approximation method for the KL expansion method (Lin et al., 2022), is employed in this study.

2) Express the stationary standard normal random field $H^{\mathrm{D}}(y)$ using Eq. (2):

$$H^{\mathrm{D}}(y) = \boldsymbol{R}^{\mathrm{T}} \boldsymbol{\Lambda}^{1/2} \boldsymbol{\xi} \tag{2}$$

where $\boldsymbol{\xi}$ is a $M \times 1$ vector whose elements are independent standard normal random variable ξ_i; $\boldsymbol{\Lambda}^{1/2}$ is a $M \times M$ diagonal matrix whose diagonal elements are $\lambda_i^{1/2}$, and λ_i are arranged in a descending order; $\boldsymbol{R}$ is a $M \times 1$ vector whose elements are $\phi_i(y)$ corresponding to λ_i; and the superscript 'T' represents the transpose of a matrix.

3) Calculate the non-normal and non-stationary random field random field $H(y)$ based on $H^{\mathrm{D}}(y)$ using Eq. (3):

$$H(y) = F^{-1}[\Phi(H^{\mathrm{D}}(y))] \tag{3}$$

where $F^{-1}(\cdot)$ is the inverse cumulative distribution function; and $\Phi(\cdot)$ is the cumulative standard normal distribution function. Obviously, a random field that obeys any probability distribution can be realized by the above process. Among them, the normal random field is most widely used in geotechnical engineering, i.e., $H(y) = \mu_H(y) + \sigma_H(y)H^{\mathrm{D}}(y)$, where $\mu_H(y)$ and $\sigma_H(y)$ are the mean and standard deviation of $H(y)$, respectively. Note that if a random field has a constant mean and standard deviation, this random field is a stationary random field.

2.2 FORM based on the iHLRF–*x*

For a limit state function (LSF) $G(\boldsymbol{X})$, calculation of the reliability index β in the original space ($\boldsymbol{X}$–space) can be expressed as follows (Low and Tang, 2004):

$$\beta = \min\left(\sqrt{(\boldsymbol{x}-\boldsymbol{\mu}^{\mathrm{N}})^{\mathrm{T}}(\boldsymbol{\sigma}^{\mathrm{N}})^{-1}\boldsymbol{\rho}^{-1}(\boldsymbol{\sigma}^{\mathrm{N}})^{-1}(\boldsymbol{x}-\boldsymbol{\mu}^{\mathrm{N}})}\right) = \sqrt{(\boldsymbol{x}^{*}-\boldsymbol{\mu}^{\mathrm{N}})^{\mathrm{T}}(\boldsymbol{\sigma}^{\mathrm{N}})^{-1}\boldsymbol{\rho}^{-1}(\boldsymbol{\sigma}^{\mathrm{N}})^{-1}(\boldsymbol{x}^{*}-\boldsymbol{\mu}^{\mathrm{N}})}\ \text{,subject to } G(\boldsymbol{X}) = 0 \tag{4}$$

where $\boldsymbol{x}$ is a realization of random variable vector $\boldsymbol{X}$; $\boldsymbol{\mu}^{\mathrm{N}}$ and $\boldsymbol{\sigma}^{\mathrm{N}}$ are the equivalent normal mean vector and standard deviation diagonal matrix at $\boldsymbol{x}$, respectively, which can be calculated by the Rackwitz–Fiessler transformation (Rackwitz and Flessler, 1978); $\boldsymbol{\rho}$ is the correlation matrix of random variables; and $\boldsymbol{x}^*$ is the design point located on the LSF $G(\boldsymbol{X})$. It is difficult to solve for the reliability index β according to the optimization problem shown in Eq. (4) because the LSF $G(\boldsymbol{X})$ in the $\boldsymbol{X}$–space is often a nonlinear implicit function. Therefore, the iHLRF–*x* is employed to calculate β, where the design point $\boldsymbol{x}^*$ can be recursively computed using Eqs. (5)–(6):

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \theta_k \boldsymbol{d}_k \tag{5}$$

$$\boldsymbol{d}_k = \boldsymbol{\mu}_k^{\mathrm{N}} + \frac{[\nabla G(\boldsymbol{x}_k)^{\mathrm{T}}(\boldsymbol{x}_k - \boldsymbol{\mu}_k^{\mathrm{N}}) - G(\boldsymbol{x}_k)]\boldsymbol{\sigma}_k^{\mathrm{N}}\boldsymbol{\rho}\boldsymbol{\sigma}_k^{\mathrm{N}}\nabla G(\boldsymbol{x}_k)}{\nabla G(\boldsymbol{x}_k)^{\mathrm{T}}\boldsymbol{\sigma}_k^{\mathrm{N}}\boldsymbol{\rho}\boldsymbol{\sigma}_k^{\mathrm{N}}\nabla G(\boldsymbol{x}_k)} - \boldsymbol{x}_k \tag{6}$$

where $\boldsymbol{x}_k$ and $\boldsymbol{x}_{k+1}$ are the kth and (k+1)th iteration point in the $\boldsymbol{X}$–space, respectively; $G(\boldsymbol{x}_k)$ and $\nabla G(\boldsymbol{x}_k)$ are the values of the LSF $G(\boldsymbol{X})$ and its gradient vector evaluated at $\boldsymbol{x}_k$, respectively; $\boldsymbol{d}_k$ is the search direction at $\boldsymbol{x}_k$; and θ_k is the step size ($0 < \theta_k \leq 1$), which can be determined by the Armijo rule:

$$\theta_k = \max_s \left(b^s \mid m(\boldsymbol{x}_k + b^s \boldsymbol{d}_k) - m(\boldsymbol{x}_k) \leq ab^s \left\langle \nabla m(\boldsymbol{x}_k),\ \boldsymbol{d}_k \right\rangle\right) \tag{7}$$

where $a \in (0, 1/2)$ and $b \in (0, 1)$ are prescribed parameters ($a = 0.2$ and $b = 0.5$ is usually applied to the recursive computation); s is an integer for optimal solution, which is set to cycle from 0 to $s_{\max}$, and $s_{\max}$ is equal to 3−5 in practice; and $m(\boldsymbol{x})$ and $\nabla m(\boldsymbol{x})$ are the merit function and its gradient vector in the $\boldsymbol{X}$–space, respectively:

$$m(\boldsymbol{x}) = (\boldsymbol{x}-\boldsymbol{\mu}^{\mathrm{N}})^{\mathrm{T}}(\boldsymbol{\sigma}^{\mathrm{N}})^{-1}\boldsymbol{\rho}^{-1}(\boldsymbol{\sigma}^{\mathrm{N}})^{-1}(\boldsymbol{x}-\boldsymbol{\mu}^{\mathrm{N}})/2 + c|G(\boldsymbol{x})|;\ \nabla m(\boldsymbol{x}) = (\boldsymbol{\sigma}^{\mathrm{N}})^{-1}\boldsymbol{\rho}^{-1}(\boldsymbol{\sigma}^{\mathrm{N}})^{-1}(\boldsymbol{x}-\boldsymbol{\mu}^{\mathrm{N}}) + c\,\mathrm{sgn}(G(\boldsymbol{x}))\nabla G(\boldsymbol{x}) \tag{8}$$

where $c \geq 100$ suffices for most engineering problems; and sgn(·) is the sign function. The Armijo rule is equivalent to applying a shrinkage factor θ_k on the search direction $\boldsymbol{d}_k$ until the merit function is sufficiently reduced.

For an implicit LSF whose value is calculated by numerical simulation, the gradient vector $\nabla G(\boldsymbol{x})$ of the LSF is usually estimated by finite difference method (FDM), where the difference step size has an important effect on the gradient vector. For random variable vector $\boldsymbol{X}$ with a specific meaning, the difference step size is easier to choose in the $\boldsymbol{X}$–space than in the standard normal space ($\boldsymbol{U}$–space).

2.3 KL–FORM–*x*

By defining $\boldsymbol{\xi}' = \boldsymbol{\Lambda}^{1/2}\boldsymbol{\xi}$, the FORM based on the KL expansion method can be performed to the $\boldsymbol{X}$–space (called KL–FORM–*x*). In the KL–FORM–*x*, the basic variable vector $\boldsymbol{\xi}'$ is a $M \times 1$ random variable vector whose elements ξ_i' are independent normal random variables with zero mean and $\lambda_i^{1/2}$ standard deviation. From the principle of the KL expansion method, the eigenfunctions $\phi_i(y)$ are of the same order of magnitude and $\xi_i' = \lambda_i^{1/2}\xi_i$ are considered as the weight of the eigenfunctions $\phi_i(y)$. This means that ξ_i' have greater numerical stability as random variables than ξ_i because difference step sizes are not scaled up or down by $\lambda_i^{1/2}$. Hence, the proposed KL–FORM–*x* provides greater numerical stability for the calculation of the reliability index β by considering ξ_i' as random variables. Specifically, the iterative convergence criteria (ICCs) of the KL–FORM–*x* are expressed as follows:

$$\text{ICC 1: } \left|\beta_{k+1} - \beta_k\right| < \varepsilon_\beta; \quad \text{ICC 2: } \max(\left|\boldsymbol{\xi}'_{k+1} - \boldsymbol{\xi}'_k\right|) < \varepsilon_d; \quad \text{ICC 3: } \left|G(\boldsymbol{x}_{k+1})\right| < \varepsilon_g \tag{9}$$

where ε_β, ε_d, and ε_g are the convergence criteria for the reliability index β, the design point $\boldsymbol{\xi}'^*$, and the value of the LSF, respectively. When all the ICCs are satisfied, the accuracy of reliability analysis is guaranteed. The reliability index of geo-structure model β is equal to β_{k+1} and the design point $\boldsymbol{\xi}'^*$ is equal to $\boldsymbol{\xi}_{k+1}'$.

Furthermore, the number of random variables for a random field equals to the number of KL expansion terms *M*. This means that the accuracy of random field discretization increases with *M*, but simultaneously the computational cost of the FORM also increases with *M* because the number of calculations of LSF is proportional to the number of random variables. Hence, the normalized global auto-covariance function error ε_M is used to balance the accuracy of random field discretization and the computational cost (Lin et al., 2022):

$$\varepsilon_M = \frac{1}{|\Omega|^2}\int_\Omega\int_\Omega\left|\rho(y,y') - \sum_{i=1}^{M}\lambda_i\phi_i(y)\phi_i(y')\right|\mathrm{d}y\mathrm{d}y' \approx \frac{1}{|\Omega|^2}\sum_{j=1}^{N_\mathrm{e}}\sum_{j'=1}^{N_\mathrm{e}}\left|\rho(y_{\mathrm{c}j}, y_{\mathrm{c}j'}) - \sum_{i=1}^{M}\lambda_i\phi_i(y_{\mathrm{c}j})\phi_i(y_{\mathrm{c}j'})\right|L_jL_{j'} \tag{10}$$

where $|\Omega|$ is the area of random field domain for geo-structure model; $y_{\mathrm{c}j}$ and $y_{\mathrm{c}j'}$ are the coordinates of *j*th grid centroid for the geo-structure model; and L_j is the length of the *j*th grid ($j = 1, 2, \ldots, N_\mathrm{e}$; and N_e is the number of grids of geo-structure model). The suitable number of KL expansion terms *M* can be selected by comparing ε_M with a predefined allowable discretization error ε_a, where ε_a ensures the accuracy of random field discretization. Generally, ε_a of 5% is suitable for most engineering problems.

3 Reliability sensitivity analysis based on the KL–FORM–*x*

3.1 Definition of the reliability sensitivity index

The results of reliability sensitivity analysis can be expressed using the dimensionless partial derivative of the reliability index β with respect to the statistical parameters $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ of random variables, i.e., the mean reliability sensitivity index (MRSI) $\boldsymbol{\psi}_\mu = \mathrm{diag}(\boldsymbol{\sigma})(\partial\beta/\partial\boldsymbol{\mu})$ and the standard deviation reliability sensitivity index (SDRSI) $\boldsymbol{\psi}_\sigma = \mathrm{diag}(\boldsymbol{\sigma})(\partial\beta/\partial\boldsymbol{\sigma})$, where $\boldsymbol{\psi}_\mu$ and $\boldsymbol{\psi}_\sigma$ are $n \times 1$ column vectors (*n* is the number of random variables), whose elements are the MRSI $\psi_{\mu i}$ and the SDRSI $\psi_{\sigma i}$ of the *i*th random variable, respectively, and diag(·) represents that the elements of a vector are applied to form a diagonal matrix. The MRSI $\boldsymbol{\psi}_\mu$ and the SDRSI $\boldsymbol{\psi}_\sigma$ are quantitative indexes and are beneficial to engineers to understand how, and to what extent, the reliability index of geo-structures changes with perturbations of random variables, and to know how changes in the material properties influence the reliability of geo-structure model under design.

Obviously, the MRSI $\boldsymbol{\psi}_\mu$ and the SDRSI $\boldsymbol{\psi}_\sigma$ for random variable vector $\boldsymbol{\xi}'$ have no physical meaning because $\boldsymbol{\xi}'$ is only a virtual random variable vector for the discretization of a random field in the KL expansion method and it is not a material parameter. Hence, by replacing the statistical parameters $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ of random variable vector $\boldsymbol{\xi}'$ with the statistical parameters $\mu_H(y)$ and $\sigma_H(y)$, where $\mu_H(y)$ and $\sigma_H(y)$ represent the mean and the standard deviation of a random field, respectively, the MRSI and the SDRSI can be redefined using Eq. (11):

$$\psi_\mu(y) = \sigma_H(y)[\partial\beta/\partial\mu_H(y)]; \quad \psi_\sigma(y) = \sigma_H(y)[\partial\beta/\partial\sigma_H(y)] \tag{11}$$

Because the calculation of reliability sensitivity index is based on the KL–FORM–*x*, the proposed reliability sensitivity analysis method is called RSA–KL–FORM–*x*. $\psi_\mu(y)$ and $\psi_\sigma(y)$ are functions of the coordinates *y* in a random field domain, so the RSA–KL–FORM–*x* is also called region reliability sensitivity analysis method. The important regions can be made up of locations with larger absolute values of the MRSI $\psi_\mu(y)$ or the SDRSI $\psi_\sigma(y)$, which is helpful for risk recognition and management in engineering practice.

3.2 Calculation of the reliability sensitivity index

Taking the MRSI as an example, the $\psi_\mu(y)$ can be rewritten based on the chain rule as follows:

$$\psi_\mu(y)=\sigma_H(y)\frac{\partial\beta}{\partial\mu_H(y)}=\sigma_H(y)\left[(\frac{\partial\beta}{\partial\boldsymbol{\xi}'^*})^{\mathrm{T}}\frac{\partial\boldsymbol{\xi}'^*}{\partial H^{\mathrm{D}}(y)^*}\frac{\partial H^{\mathrm{D}}(y)^*}{\partial\mu_H(y)}\right]^{\mathrm{T}} \tag{12}$$

where $H^{\mathrm{D}}(y)^*$ is the realization of the stationary standard normal random field for the design point $\boldsymbol{\xi}'^*$; and $\partial\beta / \partial\boldsymbol{\xi}'^*$, $\partial\boldsymbol{\xi}'^* / \partial H^{\mathrm{D}}(y)^*$, and $\partial H^{\mathrm{D}}(y)^* / \partial\mu_H(y)$ are calculated as follows.

1) $\partial\beta / \partial\boldsymbol{\xi}'^*$: The partial derivative of the reliability index β with respect to the design point $\boldsymbol{\xi}'^*$ is expressed as:

$$\frac{\partial\beta}{\partial\boldsymbol{\xi}'^*}=\frac{\partial\beta}{\partial(\boldsymbol{\Lambda}^{1/2}\boldsymbol{\xi}^*)}=\boldsymbol{\Lambda}^{-1/2}\frac{\partial\sqrt{\boldsymbol{\xi}^{*\mathrm{T}}\boldsymbol{\xi}^*}}{\partial\boldsymbol{\xi}^*}=\frac{\boldsymbol{\Lambda}^{-1/2}\boldsymbol{\xi}^*}{\beta}=\frac{\boldsymbol{\Lambda}^{-1}\boldsymbol{\xi}'^*}{\beta} \tag{13}$$

where $\partial\beta / \partial\boldsymbol{\xi}'^*$ is a $M \times 1$ vector, whose elements $\lambda_i^{-1}\xi_i'^* / \beta$ represents the effect of a small change at the design point $\xi_i'^*$ on the reliability index β, and $\boldsymbol{\xi}^*$ is the design point in the $\boldsymbol{U}$–space.

2) $\partial\boldsymbol{\xi}'^* / \partial H^{\mathrm{D}}(y)^*$: Based on Eq. (2) and the Moore–Penrose inverse, the partial derivative of the design point $\boldsymbol{\xi}'^*$ with respect to the stationary standard normal random field $H^{\mathrm{D}}(y)$ is expressed as:

$$\frac{\partial\boldsymbol{\xi}'^*}{\partial H^{\mathrm{D}}(y)^*}=\frac{\partial[(\boldsymbol{R}^{\mathrm{T}})^{\dagger}H^{\mathrm{D}}(y)^*]}{\partial H^{\mathrm{D}}(y)^*}=\frac{\partial[\boldsymbol{R}H^{\mathrm{D}}(y)^*]}{\partial H^{\mathrm{D}}(y)^*}=\boldsymbol{R} \tag{14}$$

where the superscript '†' represents the Moore–Penrose inverse of a matrix which is different from the inverse of the matrix for a square matrix. The partial derivative $\partial\boldsymbol{\xi}'^* / \partial H^{\mathrm{D}}(y)^*$ maps the relationship between $\boldsymbol{\xi}'^*$ without physical significance and infinite number of random variables $H^{\mathrm{D}}(y)^*$.

3) $\partial H^{\mathrm{D}}(y)^* / \partial\mu_H(y)$: Based on Eq. (3) and the principle of derivative of inverse function, the partial derivative of the $H^{\mathrm{D}}(y)$ with respect to the mean of random field $\mu_H(y)$ is expressed as:

$$\frac{\partial H^{\mathrm{D}}(y)^*}{\partial\mu_H(y)}=\frac{\partial\Phi^{-1}[F(H(y)^*)]}{\partial\mu_H(y)}=\frac{\partial F(H(y)^*)/\partial\mu_H(y)}{\phi\{\Phi^{-1}[F(H(y)^*)]\}}=\chi_\mu(y) \tag{15}$$

where $H(y)^*$ is the realization of the random field for the design point $\boldsymbol{\xi}'^*$; $F(\cdot)$ is the cumulative distribution function of the non-normal and non-stationary random field $H(x)$; $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of standard normal distribution; $\phi(\cdot)$ is the probability density function of standard normal distribution. Obviously, the partial derivative $\partial H^{\mathrm{D}}(y)^* / \partial\mu_H(y)$ adds the specific information of the random field to the reliability sensitivity analysis. Therefore, the $\psi_\mu(y)$ can be further rewritten from Eq. (12) as:

$$\psi_\mu(y)=[\sigma_H(y)\chi_\mu(y)\boldsymbol{R}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}\boldsymbol{\xi}'^*]/\beta \tag{16}$$

Similar to the derivation of Eq. (16), the $\psi_\sigma(y)$ can be expressed as:

$$\psi_\sigma(y)=[\sigma_H(y)\chi_\sigma(y)\boldsymbol{R}^{\mathrm{T}}\boldsymbol{\Lambda}^{-1}\boldsymbol{\xi}'^*]/\beta \tag{17}$$

where $\chi_\sigma(y) = \partial H^{\mathrm{D}}(y)^* / \partial\sigma_H(y)$. When a random field obeys normal distribution, $\chi_\mu(y) = 1 / \sigma_H(y)$ and $\chi_\sigma(y) = (H(y)^* - \mu_H(y)) / \sigma_H(y)^2$. Furthermore, for a specific geo-structure model with N_{e} grids, the coordinate $y_{\mathrm{c}j}$ can be substituted into $\chi_\mu(y)$ and $\chi_\sigma(y)$ to obtain the $N_{\mathrm{e}} \times 1$ column vectors $\chi_\mu(\boldsymbol{y}_{\mathrm{c}})$ and $\chi_\sigma(\boldsymbol{y}_{\mathrm{c}})$. From the above analysis, the values of the MRSI and the SDRSI in the proposed RSA–KL–FORM–*x* rely on the values of the reliability index β, the design point $\boldsymbol{\xi}'^*$, the eigenvalues λ_i, and eigenfunctions $\phi_i(y)$ obtained by the KL–FORM–*x*. The accuracy of the MRSI and the SDRSI is determined by random field discretization and reliability analysis whose accuracy have been ensured in Section 2.3. Because the JLG method is employed in this study, a larger number of orthogonal basis function terms N is recommended, which can obtain efficiently high precision eigenvalue λ_i and eigenfunctions $\phi_i(y)$ and reduce M for the same ε_{a} (Lin et al., 2022). Meanwhile, compared with the computational cost of the KL–FORM–*x*, the RSA–KL–FORM–*x* can be performed efficiently without additional time-consuming calculations of the LSF, where dozens to hundreds of the calculations of the LSF are sufficient to complete KL–FORM–*x*. Hence, the calculations of $\psi_\mu(y)$ and $\psi_\sigma(y)$ have high computational efficiency.

4 An illustrative example of shallow foundation

4.1 Numerical model of the shallow foundation

An undrained saturated shallow foundation is modelled by the finite difference software *FLAC2D*. As shown in Fig. 1(a), the width and height of the shallow foundation are 2 m and 1 m, respectively, and the width and height of the soil model is 10 m and 6 m, respectively. The top of the shallow foundation is loaded vertically downward with a uniform load q = 600 kPa. The left and right boundaries of the soil model are fixed only in the horizontal direction, while the bottom boundary is fixed in both horizontal and vertical directions.

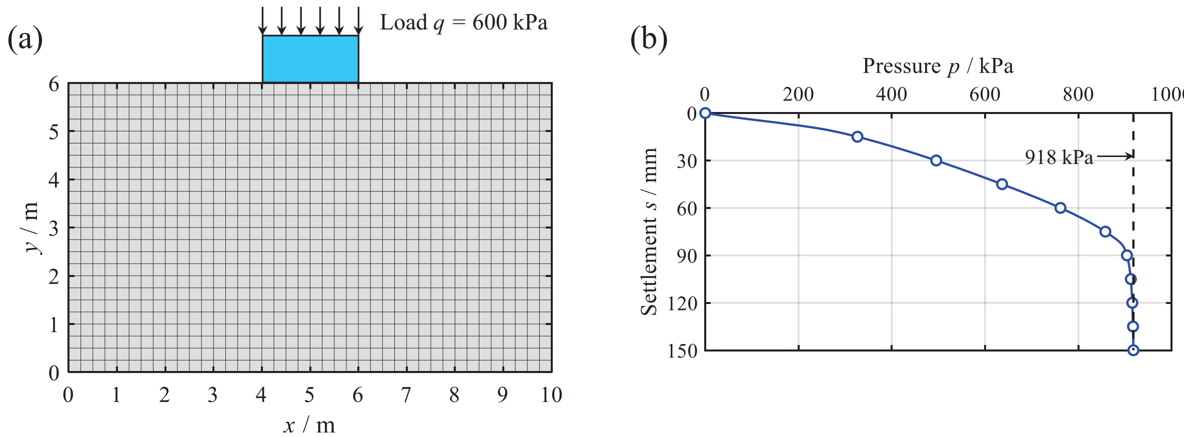

Figure 1. Geometry parameters of the shallow foundation and the p–s curve when c_s is equal to the mean of random field.

In this example, the cohesion of soil c_s is regarded as stationary normal random field, whose mean and standard deviation are 50 kPa and 15 kPa, respectively. The ACF of c_s is shown in Eq. (1), where the ACD θ of c_s is 1 m. The soil model is divided into 960 uniform grids, whose vertical direction is divided into 24 layers for realization of random field (i.e., the thickness of each layer L_j = 0.25 m, j = 1, 2, …, 24). The other strength and stiffness parameters of the soil model are constant, i.e., angle of friction ϕ_s = 20°, destiny ρ_s = 1600 kg / m^3, elastic modulus E_s = 60 MPa, and poisson's ratio v_s = 0.3. The strength and stiffness parameters of the shallow foundation are also constant, i.e., destiny ρ_f = 2500 kg / m^3, elastic modulus E_f = 20,000 MPa, and poisson's ratio v_f = 0.2. To attach the shallow foundation and soil model, the cohesion and angle of friction of the interface are set to half of those of the soil model. When c_s is equal to the mean of random field, the deformation process of soil is shown in Fig. 1(b). The settlement s of the soil at the center of the foundation base increases with the pressure p. When p increases to the ultimate bearing capacity p_u (i.e., the inflection point in the p–s curve), the settlement s increases sharply, which means that the soil is damaged. In this study, to avoid looking for inflection point, the bearing capacity when the settlement of the soil reaches 150 mm is defined as the ultimate bearing capacity p_u, which is almost consistent with the bearing capacity at the inflection point. Hence, in this situation, p_u is equal to 918 kPa.

4.2 Results of reliability analysis and reliability sensitivity analysis

A LSF of the shallow foundation about bearing capacity is expressed as:

$$Z = p_u(\boldsymbol{X}) - q = p_u(\boldsymbol{\xi}') - q \tag{18}$$

According to the procedure of the RSA–KL–FORM–x, the KL expansion method should be firstly performed. The number of orthogonal basis function terms N = 41 is selected in this example. The discretization error ε_M for different number of KL expansion terms M is shown in Fig. 2. The discretization error ε_M decreases with M, which means that the accuracy of random field increases with M. The optimal value of M is 5 according to the criterion of $\varepsilon_M \le \varepsilon_a$ = 5%. In reliability analysis, ε_β, ε_d, and ε_g are set as 0.01 for the accuracy of the reliability index and design point in all cases. As shown in Fig. 2, the reliability index β decreases with M and stabilizes, which means that the fluctuations of M > 5 have only a slight effect on β. This phenomenon also verifies the correctness of the random field discretization and reliability analysis. When M is equal to 5, the reliability index β of the shallow foundation is 1.747. Fig. 3 shows the eigenvalues λ_1–λ_5 (Fig. 3(a)) and eigenfunctions $\phi_1(y)$–$\phi_5(y)$ (Fig. 3(b)), where λ_i decreases with i, while the fluctuation of $\phi_i(y)$ increases with i.

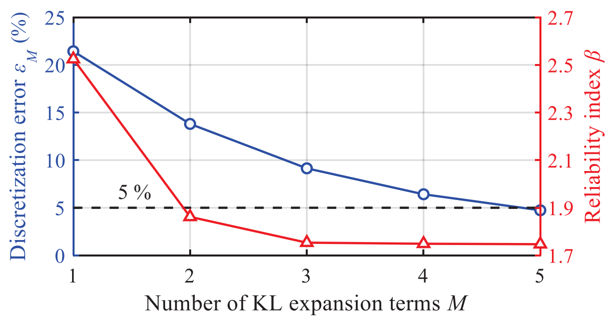

Figure 2. Discretization error ε_M and reliability index β for different number of KL expansion terms M.

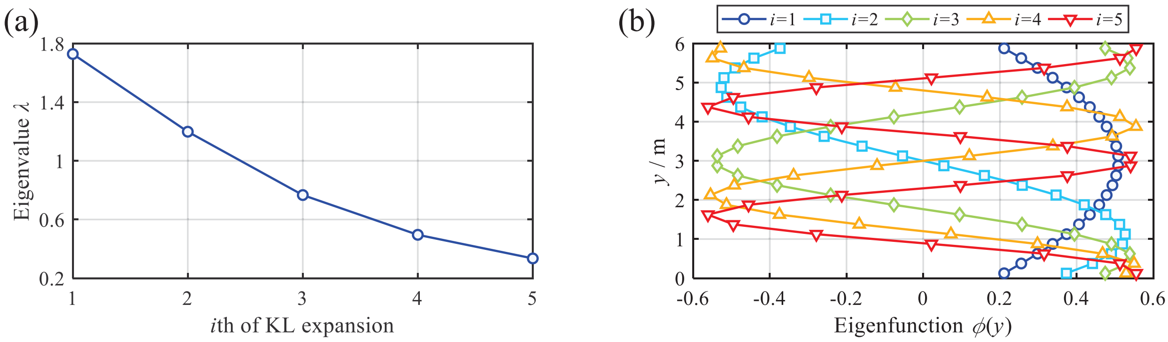

Figure 3. Eigenvalues and eigenfunctions for the random field of the cohesion of soil c_s.

After reliability analysis is performed, the results of reliability sensitivity analysis are calculated and shown in Fig. 4. The MRSI and the SDRSI moves from the middle to the top of the soil model with the increase of M and tend to stabilize, which means that $\varepsilon_a = 5\%$ satisfies the requirements of reliability sensitivity analysis. The 3–6 m range of the soil model has a very important influence on the reliability index β. Meanwhile, the location of 5 m shows that c_s has the greatest influence on β. These phenomena are consistent with the observations of Li et al. (2015) and Lo and Leung (2017), which verifies the validity of the proposed method. This means that if the cohesion of soil c_s is only regarded as a random variable, it is difficult to obtain a specific and economical solution for engineers. Meanwhile, ignoring spatial variability of soil cannot meet the actual conditions of the engineering. The important region can be considered as region for soil reinforcement, and the mean and standard deviation of random field have opposite effects on β, which means that improving the reliability of the shallow foundation can be performed by increasing the mean or decreasing the standard deviation. Therefore, reliability sensitivity analysis takes the treatment of geotechnical engineering from qualitative level to quantitative levels. From the perspective of computational efficiency, M of 5 is the maximum of number of random variables in this study. However, only 57 calculations of the LSF are required in reliability analysis, and reliability sensitivity analysis does not need to additionally calculate the LSF. Hence, the proposed method has high computational efficiency.

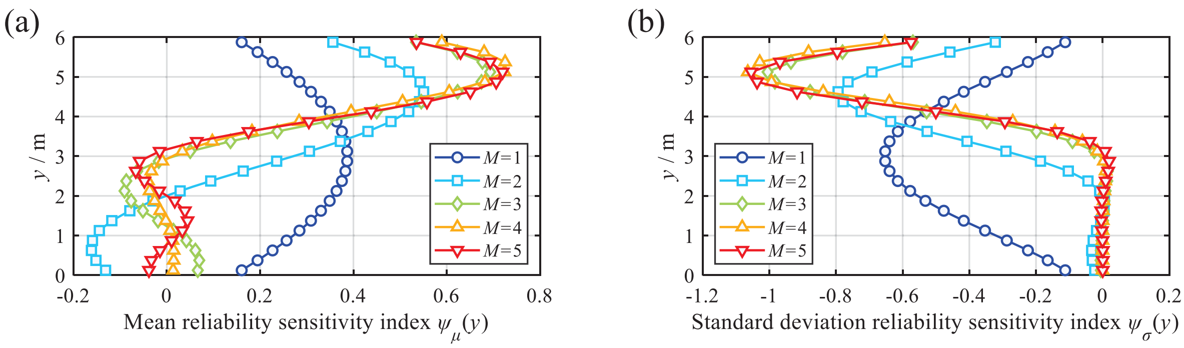

Figure 4. Mean and standard deviation sensitivity index (MRSI and SDRSI) for different number of KL expansion terms M.

5 Summary

A reliability sensitivity analysis method considering spatial variability of soil (called RSA–KL–FORM–x) is proposed in this study. In the RSA–KL–FORM–x, the Karhunen–Loève (KL) expansion method is employed to realize the random field and the first–order reliability method (FORM) is employed to perform reliability analysis, where the mean reliability sensitivity index (MRSI) $\psi_\mu(y)$ and the standard deviation reliability sensitivity index (SDRSI) $\psi_\sigma(y)$ are functions of the coordinates in random field domain. Hence, the important regions of a geo-structure model in reliability analysis can be effectively identified. An example of shallow foundation with spatially various cohesion of soil shows that the RSA–KL–FORM–x can identify important regions, which can provide a guidance for the treatment of geotechnical engineering. However, the proposed method is only applied in a one-dimensional random field in this study, so its applicability of multiple multi-dimensional random fields considered in complex geotechnical engineering problems should be further extended and investigated in future.

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