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Efficient and Conservative Estimation of Failure Probability of Strip Footing on Spatially Variable Soil Using Random Finite Element Limit Analysis

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Abstract: The present work deals with a probabilistic analysis of the bearing capacity of shallow foundation on spatially variable cohesive-frictional soils. The spatial variability of soil parameters is modelled by anisotropic random fields. The analysis of both the upper and lower bound of the bearing capacity is carried out using a random finite element limit analysis. The analyses carried out show that the results in the lower bound provide conservative estimation of the results in the mean value, and the upper bound provide conservative estimation of the standard deviation of bearing capacity. The combined approach, consisting of describing the bearing capacity using the probability distribution with the mean value obtained from the lower bound approach and the standard deviation obtained from the upper bound approach, allows for conservative, precise and efficient estimation of the probability of structural failure.

Keywords: random field; scale of fluctuation; random finite element limit analysis; strip foundation

1 Introduction

In recent decades, there has been a significant growth of interest in methods estimating the reliability of geotechnical structures. The latter is strongly associated with the modelling of spatial variability of the soil medium, which is considered the governing factor that influences the probability of failure of soil-bound structures. Nowadays a common practice is to model this variability by random fields. The need of including random fields in probabilistic analysis of geotechnical structures was even directly mentioned in annex D of the newest edition of the ISO 2394 (2015) code dedicated to reliability analysis.
A particularly popular approach for reliability assessment of geotechnical structures is the random finite element method (RFEM, Griffiths and Fenton, 1993; Fenton and Griffiths, 2008). This technique utilizes random fields to model spatial variability of soil parameters. It consists of sequential mapping the values for large number of random fields realisations on part of the finite elements mesh, representing the soil medium in a numerical model of boundary value problem. The probabilistic analysis is performed within Monte-Carlo framework: the probability of failure based on assumed limit state function can be found by statistical analysis of the results of all Monte Carlo simulations (MCSs). The important advantage of this method is its universality, which enables probabilistic modelling of virtually any geotechnical structure that can be modelled with finite elements (FE). So far, it has been used to estimate the probability of failure of strip foundations (e.g. Fenton and Griffiths 2008, Pieczyńska-Kozłowska et al. 2015), slopes (e.g. Griffths & Fenton 2004), diaphragm walls (Sert et al., 2016, Kawa et al. 2021) as well as many other geotechnical structures. On the other hand, an undoubted disadvantage of this method is the long computation time that results from the computation time of a single MCS. With a typical number of N=1000 simulations (which is still too small to credibly estimate the probability of failure in case of ultimate limit state) and the average difficulty of the problem considered, the total computation time on a modern PC is between one and several days. Such a high computational cost significantly limits the use of the method in engineering practice. For this reason, more efficient alternatives, which are based, e.g., on the quasi-analytical kinematic method (e.g. Chwała 2019) or limit equilibrium method (e.g. Liu et al., 2019), are still being sought.
One of the alternatives to RFEM, more efficient approach for modelling spatially variable soils, is random finite element limit analysis (RFELA). It is a modification of RFEM, which utilizes finite element limit analysis (FELA, Lyamin and Slone 2002a & Lyamin and Slone 2002b) instead of typical FE method (FEM) to solve individual simulations. FELA allows to determine rigorous estimation of both lower and upper bounds of bearing capacity as a constrained convex optimization problem, which is a much faster approach than traditional FEM. With the

additional use of adaptive meshing, these estimates, in successive iterations, become quite precise. Adaptive and nonadaptive algorithms of RFELA have been used for probabilistic modelling of foundations (eg. Simoes et al.,, 2014; Ali et al. 2016), slopes (Ali et al., 2017) and some other geotechnical structures.

The present work is another attempt to apply the RFELA to problem of the bearing capacity of strip foundation. The Coulomb Mohr cohesive-frictional soil is analysed. Both friction and cohesion are modelled by anisotropic random fields generated with Fourier series method (FSM, Jha and Ching 2012). The lower and upper bounds of bearing capacity are determined using FELA formulations proposed by Lyamin and Slone (2002) and Krabbenhoft et al. (2005), respectively. These formulations were implemented by the authors in Matlab code. In each realization, the solutions for the lower and upper bounds were obtained using the commercial solver MOSEK. The influence of fluctuation scales on the probabilistic characteristics of bearing capacity assessments was investigated. As shown, the results of the lower bound can be used as a fairly precise and probably conservative estimation of the bearing capacity. To ensure conservatism in estimation, a combined approach was proposed that uses both upper and lower bound results and provides only slightly lower values of allowable load for the same value of failure probability.

2 Modelling of soil parameters by random fields

In the present work, two soil parameters, ie cohesion c and the angle of internal friction φ were modelled by weakly stationary, anisotropic random fields. The fields were assumed to be not cross-correlated. Individual realizations of the fields were generated using the FSM (Jha and Ching, 2012). In the FSM, the field is defined as a continuous function represented by a finite (truncated) number of Fourier series terms with random coefficients, which allows one to read the field values for individual realisations at an arbitrary point of the domain. This enables iterative adoption of FELA mesh within a single field realization (adaptive FELA). Such adoption significantly improves the FELA results, narrowing the gap between the upper and lower bound solutions.

The 2D plane strain problem was analysed. The autocorrelation function for both fields was assumed to be of Gaussian type:

$$\rho(\tau_x, \tau_z) = \exp\left\{-\pi\left[\left(\frac{\tau_x}{\theta_x}\right)^2 + \left(\frac{\tau_z}{\theta_z}\right)^2\right]\right\} \tag{1}$$

where τ_x, τ_z are distances in horizontal (x) and vertical (z) directions, and θ_x, θ_z are scales of fluctuations (SOFs) for these directions, respectively.

The considered values of vertical and horizontal SOFs will be given in the following section; however, it should be noted that the smallest values used for both directions were identical, namely $\theta_x = \theta_z = 1.0$m. In the FSM method, the number of terms considered from the Fourier series describing the field increases for small values of the fluctuation scale. However, this number is much smaller for the Gaussian autocorrelation function than, for example, for the exponential function. The maximum number of Fourier series terms considered in this paper (for the mentioned values of SOFs) was app. 3700.

The parameters of the probability distributions were assumed as follows: the mean value and standard deviation for cohesion were assumed as μ_c=29.0 kPa and σ_c=7.0 kPa, respectively, and for the friction angle as μ_φ=12.41° and σ_φ=1.15°, respectively. Since the type of probability distribution was assumed to be lognormal and FSM generates only the normally distributed field, all the fields realizations for the parameter X ($X \in \{c,\varphi\}$) were actually generated for its underlying normal distribution parameters μ_Y, σ_Y derived using the following formulas:

$$\sigma_Y{}^2 = \ln\left(1 + \frac{\sigma_X{}^2}{\mu_X{}^2}\right),\ \ \mu_Y = \ln(\mu_X) - \frac{1}{2}\sigma_Y{}^2 \tag{2}$$

and then the values of this random field read for a given realization at point $\mathbf{x_0}=\{x_0, z_0\}$, $\mathrm{RF}_Y(\mathbf{x_0})$, were transformed to values of the lognormally distributed field $\mathrm{RF}_X(\mathbf{x_0})$ using the following relation:

$$RF_X(\mathbf{x}_0) = \exp(RF_Y(\mathbf{x}_0)) \tag{3}$$

3 Numerical model analysed with FELA

The considered problem is the bearing capacity of the surface strip foundation on spatially variable cohesive-frictional material. To solve the problem, FELA algorithms providing rigorous upper and lower bounds of the solution in the case of a perfectly plastic soil model and associated flow rule were employed. Both FELA formulations used, i.e., lower (Lyamin and Slone, 2002) and upper (Krabbenhoft et al. 2005), assume triangular elements, linear shape functions for the fields approximated within the elements (stresses in lower bound formulation and displacement velocities in upper bound formulation, respectively) and admissible (statically or kinematically, respectively) discontinuities of these fields between the elements. Both problems can be defined as the following optimisation problem:

$$\begin{array}{c} \text{maximize } \alpha \\ \text{subject to: } \mathbf{A\sigma} = \alpha\mathbf{p} + \mathbf{p_0},\, f(\boldsymbol{\sigma}) \le 0 \end{array} \tag{4}$$

where α is the optimized load multiplier, $\boldsymbol{\sigma}$ denotes the vector of stresses in all domain elements, $\mathbf{A}$ is the equilibrium matrix (containing also the boundary conditions), $\mathbf{p}$ and $\mathbf{p_0}$ is the optimized and constant part of the external load and f is a limit function, assumed here to be the Mohr-Coulomb one. Obtaining form (4) based on the finite element geometry of the mesh and the applied boundary conditions is not trivial. Appropriate transformations enabling its derivation for both formulations were implemented by authors in the Matlab environment.

The Mohr-Coulomb criterion in the plane strain state can be written as a second-order cone (SOC). For that reason, the above optimisation problem can be assigned to the SOC class and as such can be successfully solved by a number of commercial and noncommercial optimisation solvers including e.g. sdpt3, Sedumi, or MOSEK. The last of the mentioned seems to be particularly efficient in FELA analysis (Podlich et al., 2014). This solver was also employed in the present work.

The width of the analysed footing was assumed to be 1 m. The footing itself was assumed to be rough and weightless. The unit weight of the soil was assumed as 20 kN/m^3. As mentioned above soil friction and cohesion were described by a random fields. In fact, each triangular element was assigned a constant parameter value being the average of the cohesion and internal friction fields over the area of the element. To calculate this average, the values of the field were read in the points lying in the centres of the three sides of each element, and the mean of those reads (which corresponds to the second-order Gauss quadrature average over triangular area) was assumed as the parameter value for the element. The boundary conditions for the problem domain are presented in Fig. 1. The dimensions of the domain were assumed to be 10 m x 20 m. As verified, in none of the realisations the failure mechanism reached the side or lower boundaries of the area.

For mesh generation, the Delaunay-based mesh generator for Matlab – MESH2D (Engwirda 2014) was used. In the corners of the foundation, 'element fans' (Lyamin and Sloan 2003) consisting of 30 elements with a vertex angle of 6 ° were used. The application of such 'element fans' significantly influenced the results of the lower bound. To further decrease the difference between the lower and upper estimates a procedure for adaptive mesh refinement, which was based on the value of the plastic multiplier obtained for the upper bound approach (Muñoz et al. 2009), was adopted. In a first step, a uniform mesh (except for the elements fans and their vicinity) with an average element size of 0.5 m (app. 5500 elements) was generated and for all the elements plastic multiplier values were determined. In the second step, a finer mesh with an average element size of 0.3m was generated. However, this mesh was no longer uniform: the function of its spatially variable density resulted from the distribution of the plastic multiplier in the first step. In the subsequent steps, the 10% elements with the highest values of the plastic multiplier in the previous step were divided into four new elements. The total number of steps adopted was five: this number was a compromise between accuracy and efficiency. In the last step of the procedure, the number of elements (depending on the realization) was 9000-14000. For a sample realization of fields, the refinement procedure is shown in Fig. 1.

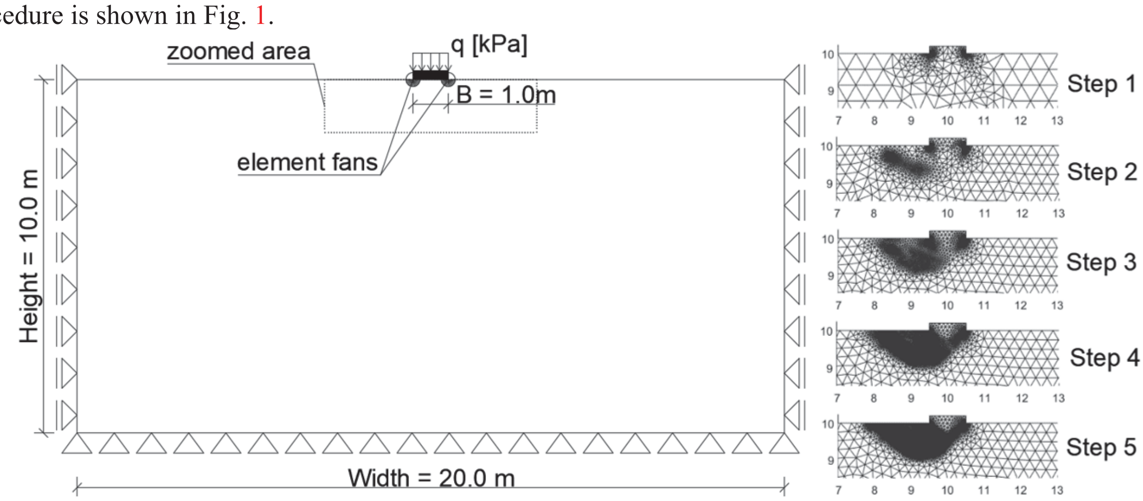

Figure 1. Geometry of the model and mesh plot in the zoomed area for five steps of the adaptive mesh refinement procedure in one realization

A total of five cases with different values of horizontal SOF, i.e. 1 m,,, 2 m, 5 m, 10 m and 100 m were analysed. In each of these cases, the vertical SOF was assumed to be 1 m. The fields modelling φ and c were generated for the same values of SOFs. In each of the cases analysed, the calculation of both upper and lower bounds of the bearing capacity was performed for N=1000 MCSs. The computation time for a single series (1000) of calculations was about 5 hours. The results of the analysis are presented in the next section.

4 Results

The results of the analysis for all the analysed cases are presented in Fig. 2 in the form of histograms and fitted probability density functions (PDFs) of estimated normal distributions for both upper and lower bounds of bearing capacity. In individual columns, different cases of assumed horizontal SOF are presented, whereas the results for the successive steps of adaptive refinement are shown in subsequent rows. Both histograms and PDFs for lower bound are shown in the figure with blue colour. Analogously results for upper bound are shown with red colour.

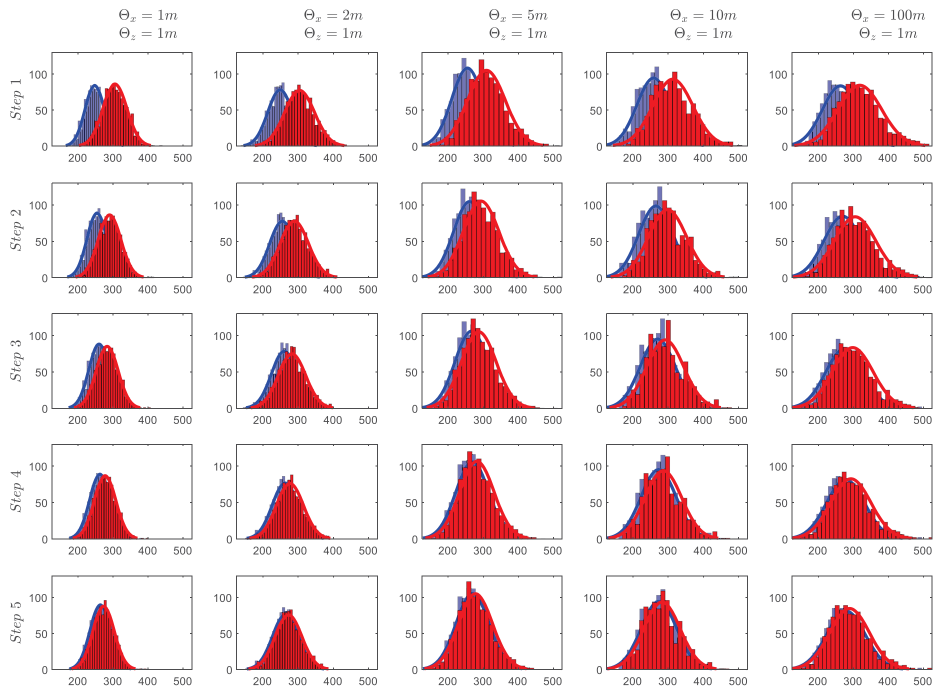

Figure 2. Upper (red lines) and lower (blue lines) bounds of the bearing capacity of a shallow foundation

As seen, the results for different cases are characterized by relatively similar mean values. On the other hand, it is clearly visible that standard deviations (and thus the range of histogram values and the 'width' of PDF fit) increases for grater values of the horizontal SOF. Also, the influence of mesh adaptively is visible in the individual columns in the figure. After five iterations for all the analysed cases, the results for lower and upper bounds seem to be practically identical.

The estimated values of the mean, standard deviation and coefficient of variation of the upper and lower bound of the bearing capacity of the analyzed foundation for all the adaptive refinement procedure are shown in Fig. 3. The results for different horizontal SOF θ_x are shown again in different columns. Also, the lower and upper bounds are again colored blue and red, respectively. As seen, although the mean of both the lower and upper bounds of the bearing capacity slightly increases with increasing value of horizontal SOF, the relative differences between the values obtained for the fifth step in the case of minimum (1m) and maximum (100 m) horizontal SOF are not greater than 7%. The relative differences between the standard deviations are much greater, up to 40%.

All the characteristics obtained for the final fifth step of the adaptive procedure, both for the lower and upper bounds of the bearing capacity, are summarized in Table 1. These values confirm that in the last step the characteristics for upper and lower bounds are very similar. The differences between mean values and standard deviations do not exceed 4%.

Based on the distribution fits (for both upper and lower bounds) and the assumed typical value of probability of failure 7.23 x 10^{-5} (corresponding to the reliability index β=3.8) the allowable design loads for the foundation were calculated and are shown in Table 2. As seen, the value obtained using the lower bound based distribution in all the cases is lower than the values obtained using the upper bound based distribution. This means that probably safe design values can be estimated from lower bound results. However, it is worth noting that the standard deviation of the exact solution (which lies between the lower and upper bounds) is probably larger than that obtained for the lower bound. Furthermore, the results were estimated only based on N=1000 realizations and thus the obtained fits do not allow for credible estimation of allowable loads for such low values of probability (see, e.g., Kawa et al. 2021). For this reason, a different, more conservative approach is proposed, namely estimating the allowable load value based on a new distribution with conservatively estimated (underestimated) mean value obtained from lower bound results and conservatively estimated (overestimated) standard deviation obtained from upper bound results. The results of such a mixed approach are also shown in Table 2. The application of the proposed method for the determination of the allowable values should be considered as a fairly precise and (due to the high efficiency of the FELA method) quick estimate that can be applied in engineering practice.

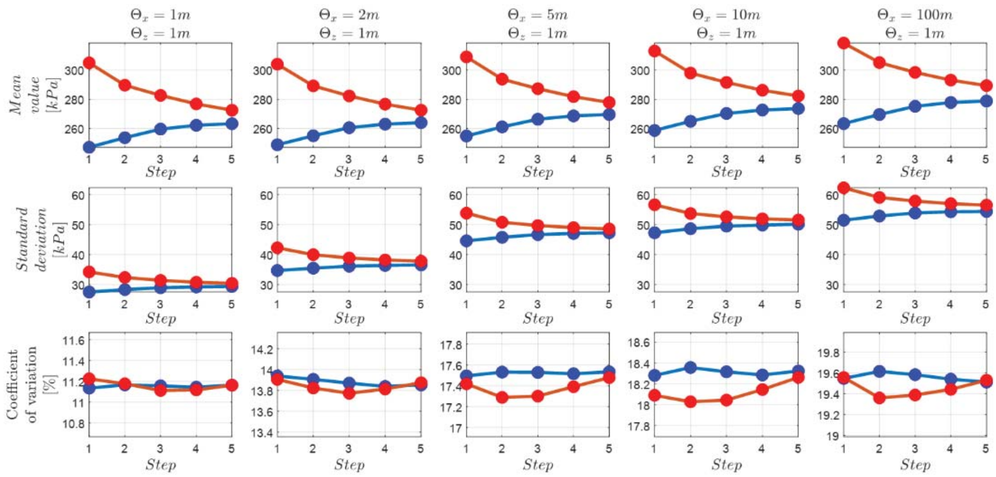

Figure 3. Diagrams of mean value, standard deviation, and coefficient of variation as a function of the step of adaptive mesh refinement procedure for the following scales of fluctuations

Table 1. The mean values, standard deviations, and coefficients of variation for the lower and upper bound of the bearing capacity estimated based on the results of the the last (5[th]) step of adaptive mesh procedure.

Scales of fluctuations	Lower bound estimation			Upper bound estimation		
	Mean value (kPa)	Standard deviation (kPa)	Coefficient of variation (%)	Mean value (kPa)	Standard deviation (kPa)	Coefficient of variation (%)
θ_x =1.0m, θ_z =1.0m	263.30	29.39	11.16	272.52	30.42	11.16
θ_x =2.0m, θ_z =1.0m	264.00	36.57	13.85	272.52	37.81	13.87
θ_x =5.0m, θ_z =1.0m	269.61	47.27	17.53	277.83	48.56	17.48
θ_x =10.0m, θ_z =1.0m	273.65	50.14	18.32	282.28	51.55	18.26
θ_x =100.0m, θ_z =1.0m	278.80	54.40	19.51	289.36	56.51	19.53

Table 2. Allowable load for β=3.8 obtained for LB FELA, UB FELA and mixed approach

Scales of fluctuations		LB FELA [kPa]	UB FELA [kPa]	MIXED [kPa]
θ_x =1.0m, θ_z =1.0m		151.60	156.92	147.70
θ_x =2.0m, θ_z =1.0m		125.04	128.85	120.32
θ_x =5.0m, θ_z =1.0m		89.97	93.29	85.08
θ_x =10.0m, θ_z =1.0m		83.11	86.39	77.75
θ_x =100.0m, θ_z =1.0m		72.07	74.60	64.05

5 Conclusions

In this study, the RFELA method was used to assess the bearing capacity of a strip foundation on spatially variable cohesive-frictional soil. The formulations proposed by Lyamin and Slone (2002) and Krabbenhoft et al. (2005) implemented by authors in Matlab, together with MESH2D (Engwirda 2014) and the MOSEK solver were used. Anisotropic random fields that model soil parameters were generated by FSM (Jha and Ching 2012). The results of the lower and upper bound of the foundation bearing capacity were estimated for five different cases of the horizontal fluctuation scale. N=1000 MCSs were analysed for each of these cases.

The analyses performed clearly show the effectiveness of the adopted mesh refinement procedure. Differences between mean values and standard deviations of the upper and lower bound bearing capacities estimated in the fifth step of mesh refinement do not exceed 4%.

The proposed approach for the estimation of the allowable value for the bearing capacity consists of using the distribution with mean value estimated by lower bound results and standard deviation estimated by upper bound results. Such a combined approach allows both the probability of failure and the allowable load for a given probability to be estimated safely and efficiently.

This study is preliminary. Currently, the authors are trying to improve the algorithm used. Some of the limitations that must be overcome are listed below.

i) In the present study, according to approach formulated by Krabbenhoft et al. (2005) for upper bound finite element limit analysis the constant strain elements were utilised. It is possible to obtain a better agreement between the lower and upper bound by using linear strain elements for upper-bound limit analysis (Makrodimopoulos and Martin, 2007). This approach is currently being tested.

ii) The reliability of design load values obtained using upper- and lower-bound results can be estimated using e.g. subset simulations (Au and Beck 2001). This could probably allow for more precise estimation of allowable load based eg. on lower bound estimation only. This is also a subject of current development.

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