Bayesian Subsurface Mapping Using CPT Data

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Abstract: Data-Driven Site Characterization aids geotechnical engineering and decision-making by producing a 3D soil parameter map of the subsurface. Unfortunately, limited site investigation data availability renders most traditional Machine Learning methods inadequate. In this paper, a framework for Bayesian Site Characterization (BaySiC) is applied on an artificial site investigation dataset. The framework aims to evaluate the statistics of the CPT measurements (cone resistance and sleeve friction) and establish their relationship, but also identify the soil heterogeneity patterns and classify the soil type at each point of the subsurface. The Bayesian framework can deal with small site investigation datasets and quantifies the prediction uncertainty with runtimes that are feasible for practical application. Essentially, the Bayesian framework maps the parameters over the subsurface on a probabilistic level. Therefore, the potency of the framework is not only judged based on traditional metrics, such as the adjacency of the mean prediction to the data, but also the likelihood attributed to the data, which is intuitively expressed by the visualization of the prediction credible intervals over the subsurface. This is shown by means of a case from a benchmark exercise.

Keywords: Site Characterization; Bayesian Inference; Subsurface Mapping, CPT measurements; Machine Learning

1 Introduction

Ideally, site characterization would give us full knowledge of the subsurface, i.e. soil layering and all relevant soil parameters. But the reality is that geotechnical engineering must find ways to merge the limited amount of data with the existent/empirical knowledge. Data-driven site characterization aids geotechnical engineering to predict soil parameter values and mapping them over the subsurface. Limited site investigation data availability renders most traditional Machine Learning methods inadequate for site characterization. However, Machine Learning approaches enabled by Bayesian techniques prove effective in such settings. Bayesian inference updates parameter distributions by combining pre-existing knowledge to data, accurate predictions, as well as quantification of the associated uncertainty. Especially the quantification of the uncertainty is of extreme importance during the decision-making process. In addition to that, identifying spatial correlation and autocorrelation patterns and producing a 3D mapping of the subsurface, can highly help understanding the problem at hands, and allow the prediction of soil parameter values (and their probability distribution) per location of the subsoil. The Bayesian Site Characterization (BaySiC) framework presented in this paper aims thus to interpret site investigation data on a probabilistic level using Bayesian inference. It can establish relationships between parameters, identify spatial correlation patterns and assess the uncertainty of parameter predictions.

Section 1 and 2 are introductory sections to the problem and the theory behind the Bayesian site characterization framework. Section 2 also presents the methodology for the subsurface mapping using Bayesian techniques, and Section 3 demonstrates the application and performance by means of the benchmark exercise presented in (Phoon et al., 2022)(Phoon et al., 2022)(Phoon et al., 2022)(Phoon et al., 2022) using CPT measurements. Finally, Section 4 presents the conclusions and recommendations.

2 Theoretical background

2.1 Bayesian updating

Bayesian inference employs Bayes’ theorem to draw conclusions on the variables (X) by conditioning on observations (ε)(Gelman et al., 2013). Bayesian inference entails the setup of a statistical model, that determines the sources of epistemic (reducible) and aleatory (irreducible) uncertainty present in the examined problem. Besides, Bayesian inference aims to reduce epistemic and provide a better description of aleatory uncertainty. In this study, Bayesian inference is performed by sampling using the Hamiltonian Monte Carlo (HMC) algorithm (Neal, 2011).

2.2 Bayesian model formulation

The benchmark exercise entails training the inference model using the CPT measurement data (observations ε) at the training locations and predicting the values of the measurements at the validation locations in a multi-layer
subsoil domain. Specifically, the CPT measurements are the cone resistance ($q_c$) and the sleeve friction ($f_s$). Moreover, the exercise requires the prediction of the layer type at the validation locations. The BaySiC framework employed in this study examines the benchmarking exercise through a Bayesian perspective. The goal is to use a statistical model that describes the observations and connects them to predictions at the test locations.

The basis of the statistical model is the assumption that the observations ($\epsilon$) at each training point can be approximated by a constant mean ($\mu$), which is a function of variable vector ($X$) and the coordinates per point (Eq. 1) (Geyer, Papaioannou and Straub, 2021), as well as a measurement error term ($\epsilon$), which in this study is assumed to be null. BaySiC models the subsurface using random fields (RF) (Vanmarcke, 2010). For this example, where RFs for 2 parameters ($q_c$ and $f_s$) have to be jointly modelled, RF modelling adopts a multivariate normal distribution, whose covariance matrix ($\Sigma$) incorporates the variance of the observed variables $q_c$ and $f_s$, the cross-correlation between them, as well as the autocorrelation due to spatial variability. Consequently, the observations can be described by the likelihood function of Eq. 2, since both $q_c$ and $f_s$ at all measurement points is a multivariate normal distribution. Even though this assumption might lead to issues such as negative values, it is adopted for compliance with the benchmark’s settings.

$$\epsilon(x, y, z) = [\epsilon_q(x, y, z), \epsilon_f(x, y, z)]^T = \mu(X, (x, y, z)) + \epsilon$$

$$L_q(X) = N(\epsilon|\mu, \Sigma)$$ (1) (2)

The statistical model uses the variable vector $X$ as the basis of the assumed behavior. It should be noted that the model follows a “lumped” layer approach. Essentially, it assumes the existence of a single layer in the subsoil domain and cannot distinguish between the different layers of the subsoil. The main random variables of $X$ are given in Eq. 3. In detail, they are: the vector of global mean per observed variable ($\mu$), the vector of standard deviation per observed variable ($\sigma$), the cross-correlation matrix ($C_{cross}$), and the vector of autocorrelation lengths per direction of the subsoil ($\theta$).

$$X = [\mu_q, \mu_f, \sigma_q, \sigma_f, C_{cross}, \theta_h, \theta_v]^T$$ (3)

For the sake of showcasing the effectiveness of the method, as well as due to the lack of context about the subsoil of the exercise, weakly informative priors are assumed for all variables of vector $X$. It should be noted that the lack of informative priors means that the influence of the likelihood function and the observations on the inference of the posteriors is expected to be greater.

Following this, some auxiliary variables are defined by the statistical model to aid the evaluation of the likelihood function. In detail, $C_{cross}$ is the cross-covariance matrix, estimated by the standard deviation vector and the cross-correlation matrix. Also, the autocorrelation matrix ($C_{auto}$) is derived by applying the distances between the training points ($d_{mat,h}$ and $d_{mat,v}$ for the horizontal and vertical directions respectively) and the vector of autocorrelation lengths in the Markov autocorrelation function (Eq. 4). Lastly, $\Sigma$ is defined as the Kronecker product of the cross-covariance and autocorrelation matrices (Eq 5), and integrates their influence in the likelihood function.

$$C_{auto}(\theta) = \exp\left(\frac{(d_{mat,h})^2}{\theta_h^2} + \frac{(d_{mat,v})^2}{\theta_v^2}\right)$$ (4)

$$\Sigma = C_{auto} \otimes C_{cross}$$ (5)

### 2.3 Posterior predictive distribution at the validation points

The points in the 3D subsurface can be distinguished between training and validation points, which belong to the training and validation locations respectively. The training points, where the observations are available, have already been used in conditioning the Bayesian model and deriving the posterior distributions of the variables, as described in paragraph 2.2. The assumption made for the likelihood function of BaySiC is expanded to the validation points. Now, the $q_c$ and $f_s$ values at the training and predictions points are assumed to follow a multivariate normal distribution with mean $\mu_\delta$ and covariance $\Sigma_\delta$, both of them being functions of the inferred variable vector $X$. The values at the training points are given, so making predictions requires conditioning the multivariate distribution to the observations at the training points. According to (Marriott and Eaton, 1984), the conditional distribution of the CPT measurements at the prediction points ($\epsilon_{p}$) given the observations at the training points ($\epsilon_{t} = \epsilon$) is a multivariate normal distribution parametrized by the mean $\mu_{\delta, |\epsilon_{p}=\epsilon}$ and the covariance $\Sigma_{\delta, |\epsilon_{p}=\epsilon}$ (Eq. 6 and Eq. 7). Eq. 8 to Eq. 10 formulate the derivation of these parameters as a function of the components of the global mean vector and covariance matrix. As a result, the conditional distribution is a function of the variables inferred by the Bayesian model and the measurements at the training points.

$$\mu_\delta = [\mu_t, \mu_v]^T$$ (6)
\[ \Sigma_x = \begin{bmatrix} \Sigma_{tt} & \Sigma_{tv} \\ \Sigma_{vt} & \Sigma_{vv} \end{bmatrix} \]

\[ (e_t | e = \varepsilon) \sim \mathcal{N}(\mu_{v|e=\varepsilon}, \Sigma_{v|e=\varepsilon}) \]

\[ \mu_{v|e=\varepsilon} = \mu_v + \Sigma_v \Sigma_{tt}^{-1} (\varepsilon - \mu_t) \]

\[ \Sigma_{v|e=\varepsilon} = \Sigma_{vv} - \Sigma_v \Sigma_{tt}^{-1} \Sigma_{tv} \]

This conditional distribution is used to determine the posterior predictive distribution at the validation points, by marginalizing over the posterior distribution of the Bayesian model’s variables \(X\) (Gelman et al., 2013). Having established that the parameters of the conditional distribution are function of the inferred variables, the posterior predictive distribution at the validation points can be expressed through the posterior sample of the Bayesian model as in Eq. 11. Samples from the posterior predictive take the form of random fields of CPT measurements, conditioned to the measurements at the training locations.

\[ P(e_p | e_t = \varepsilon) = \frac{1}{N} \sum_{i=1}^{N} \mathcal{N}(\mu_{v|e=\varepsilon}(X_i), \Sigma_{v|e=\varepsilon}(X_i)) \]

2.4 Prediction evaluation metrics

This study employs three metrics for the evaluation of the competency of the presented approach per case. Two metrics were provided by the benchmark exercise creators as common comparison ground for all methods (Phoon et al., 2022). These are the Root Mean Squared Error (RMSE) and the Identification Rate (IR). Firstly, the RMSE (Eq. 12) is a means of assessing the average distance between the cone resistance of the validation dataset and the prediction per location \(n\) points are present over the vertical profile of each validation location. Since BaySiC derives conclusions about \(X\) on a probabilistic level, the expectation of RMSE is used as the comparison metric. The RMSE means formulated in Eq. 13, where \(N\) is the number of posterior samples of \(X\). Secondly, the IR (Eq. 14) counts the points per validation location where the layer prediction, quantified by the Robertson’s Soil Behavior Type (SBT) number matches the validation dataset (Robertson, 2016). The estimation of the SBT, and thus the IR, are functions of the variable vector \(X\). The mean of IR is evaluated according to Eq. 15.

\[ \text{RMSE}(X) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (e_i - \mu_{p|e=\varepsilon}(X))^2} \]

\[ \bar{\text{RMSE}} = \int \text{RMSE}(X)p(X|e) dX = \frac{1}{N} \sum_{i=1}^{N} \text{RMSE}(X_i) \]

\[ \text{IR}(X) = \frac{1}{n} \sum_{i=1}^{n} I(X_i) \]

\[ \bar{\text{IR}} = \int \text{IR}(X)p(X|e) dX = \frac{1}{N} \sum_{i=1}^{N} \text{IR}(X_i) \]

Figure 1. Panel a) shows the locations of the training CPTs and panel b) the locations of the validation CPTs in the top view of the soil domain. Panel c) shows the stratigraphy of the subsoil. The figures are taken from the benchmark example.

3 Subsurface mapping results

3.1 Benchmark exercise description

This section demonstrates the application and performance of the BaySiC framework described in section 2 in the benchmarking exercise of (Phoon et al., 2022). The exercise focuses only on one of the available benchmark settings, which employs training set T3 and subsoil case G1. The top view of the training and validation set
formations is shown in Figure 1. As observed, validation CPT locations can exist outside the formation of training locations. The CPT profiles of all locations reach the full 10 meter depth of the subsoil domain. Moreover, the same Figure 1 presents the subsoil formation of the exercise, which consists of horizontal layers of sand, clay and silt.

The training data of the benchmark exercise is the input of the BaySiC framework. BaySiC conditions the random variables (Eq. 3) to this data and infers their posterior distributions and the posterior predictive distribution at the validation points. The latter is used to determine the expectations of the validation metrics, which indicate the performance of BaySiC.

3.2 Inference results

Inference is performed on the training set T3 and 10,000 samples are collected from the posterior. The inference time for collecting the samples is approximately 60 minutes, as performed by a laptop with an i5 processor and 8GB of RAM.

The posterior of the global means and standard deviations for the CPT measurements are presented in Figure 2. Considering that this is a lumped layer approach, the posteriors are not expected to approximate the underlying target values that generated the training data. The same figure shows the posterior distribution of the coefficient of correlation between \( q_e \) and \( f_s \) (\( \rho \)), which is the off-diagonal term of the covariance matrix. The posterior shows undeniable correlation between the observed variables. Specifically, the 5th quantile of the posterior sample is \( \rho_{0.05} = 0.37 \), while the lowest sample found is \( \rho_{min} = 0.33 \).

![Figure 2](image)

**Figure 2.** Posterior of the a) mean and b) standard deviation for the \( q_e \) and \( f_s \). The mean and standard deviation of \( f_s \) have been scaled by \( 10^{-1} \) for better visualization. Panel c) shows the posterior of the cross-correlation factor between \( q_e \) and \( f_s \).

![Figure 3](image)

**Figure 3.** Comparison of the autocorrelation values for the 5th and 95th quantiles of the posteriors of: a) \( \theta_v \) and b) \( \theta_h \).

The spatial variability of the “lumped” layer is controlled by the autocorrelation lengths. The impact of the variance of these variables is not directly clear; so Figure 3 compares the autocorrelation function plot over distance for the 5th and 95th quantiles of each \( \theta \) posterior. The difference between the quantile lines is small for the vertical direction but can be considerable for the horizontal direction. The observations are densely packed over the vertical (along the CPT profile), so characterization of \( \theta_v \) is quite accurate. On the other hand, the CPT formation appears to have considerably greater distances between the locations than \( \theta_h \), which leads to poorer inference.

3.3 Prediction results

Following, the inferred posterior is used to predict the observed variables (\( q_e \) and \( f_s \)) at the validation points and evaluate the adopted metrics, as described in paragraph 2.4. The scores of the method are exhibited in Table 1.
Table 1. Resulting mean per metric

<table>
<thead>
<tr>
<th>Metric</th>
<th>RMSE [-]</th>
<th>IR [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>1.23</td>
<td>0.81</td>
</tr>
</tbody>
</table>

Figure 4 presents the mean prediction of $q_t$ at the validation locations. The $q_t$ mean lies closely to the validation data and the conditional fields from the posterior predictive achieves an RMSE of 1.23 MPa. The expectation profile in all locations follows the $q_t$ of the clay layer closely, due to its low deviation. At some depths, the prediction profile deviates from the validation data in the sand and silt layers, which fluctuate more intensely. Although such deviations exist, the method is still able to grasp the general trend and follow the shape of the validation data per layer. Moreover, the effect of adopting a “lumped” layer approach is apparent; the prediction curve transfers smoothly between layers, because it has not been set up to comprehend the existence of multiple layers. Additionally, the credible intervals envelope the validation data over a considerable range of the profiles, while also closely capturing the validation curve shape. Lastly, a shortcoming of adopting the ‘lumped’ layer approach is that the prediction uncertainty per layer is influenced by the uncertainty of all layers. As a result, the prediction uncertainty of the clay layer is relatively large, as it is affected by the uncertainty of the sand and silt layers, which are considerably greater.

![Figure 4](image1.png)

**Figure 4.** Mean prediction of $q_t$, 95% credible interval and validation data at the validation locations.

Figure 5 demonstrates the accuracy of the layer type prediction. BaySiC achieves an average IR of 0.81. Specifically, BaySiC is competent in following recognizing the body of the layer, meaning that it can follow the main type of each formation. According to the SBT estimation for the validation data, each layer contains lenses of other material types. Although the prediction has been accurate for some lenses, it is overall not able to detect them consistently. This behavior is expected; because lenses occur as localized anomalies for the training and validation sets and hardly follow a spatial pattern.

![Figure 5](image2.png)

**Figure 5.** Layer type prediction based on the mean SBT prediction along with the validation data at the validation locations.
Lastly, it is worthwhile to mention that expectedly, the greatest RMSE and lowest IR scores are met at validationCPTsthat are located further from the training CPT formation. Greater distances from the training set reduce the horizontal autocorrelation and makes the predictions more uncertain and less accurate.

4 Conclusions

This paper presents BaySiC, a framework for Bayesian Site Characterization, by approaching a benchmarking exercise for subsurface mapping of CPT parameters provided by (Phoon et al., 2022). The goal of the exercise is to illustrate the effectiveness and efficiency of Bayesian methods in a data driven site characterization problem. The study adopts a Bayesian perspective and brings the problem to a probabilistic level, where all predictions are associated with the corresponding probability of occurrence. Naturally, the method focuses on accurate predictions through uncertainty reduction.

Firstly, a statistical model that describes the global uncertainty and spatial variability of CPT parameters has been set up. A centerpiece of BaySiC is the definition of the likelihood function, which establishes relationships between observations over a 3D subsoil domain. Bayesian inference is used to condition the variables of the statistical model on the CPT observations at the training locations of the exercise. Subsequently, a prediction step determines the parameter distributions at the validation points.

One case of the benchmark has been analyzed in the paper, to the end of demonstrating the functionality of BaySiC. The predictions of the method are competent in approximating the validation data in terms of parameter values and profile shape. Additionally, the method is in general effective in recognizing the layer type per validation point but suffers from the presence of localized anomalies. Ultimately, BaySiC has proven to be competent in deriving posterior distributions that lead to accurate and precise predictions both in terms of CPT parameter values and layer type identification.

In conclusion, the paper suggests a Bayesian standpoint to the topic of Machine learning for subsurface mapping using CPT parameters. Even with a simplified perception of the subsurface, the Bayesian model has been effective in generating an accurate map of cone resistance, sleeve friction and layer types of the subsoil. Lastly, it is noted that the method has proven to be computationally efficient, achieving runtimes that can be facilitated by project settings.

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References


