Proceedings of the 33rd European Safety and Reliability Conference (ESREL 2023)
Edited by Mário P. Brito, Terje Aven, Piero Baraldi, Marko Čepin and Enrico Zio
 Published by Research Publishing, Singapore.
doi: 10.3850/978-981-18-8071-1_P572-cd

Computing Upper Probabilities of Failure Using Optimization Algorithms Together with Reweighting and Importance Sampling

Thomas Fetz
Unit of Engineering Mathematics, University of Innsbruck, Austria. E-mail: Thomas.Fetz@uibk.ac.at

Michael Oberguggenberger
Unit of Engineering Mathematics, University of Innsbruck, Austria.
E-mail: Michael.Oberguggenberger@uibk.ac.at

The presentation addresses efficient computation of the upper probability of failure of engineering structures, when the uncertainty is modelled by a family of probability densities. We develop an algorithm significantly reducing the sample sizes required in the optimization algorithm by adopting a recursive importance sampling scheme.

Keywords: Upper probability, global optimization, importance sampling, reweighting, imprecise probability.

1. Parametrized and Upper Probabilities

Let $\{X_t\}_{t\in\mathcal{T}}$ be a family of multivariate random variables $X_t : \Omega \to D \subseteq \mathbb{R}^{d_1}$ with corresponding density functions $f_t : D \to \mathbb{R}$ which are parametrized by $t = (\tau_1, \ldots, \tau_{d_2}) \in \mathcal{T}$. Further let $h : D \to \{0, 1\}$ be an indicator function on D where $h(x) = 1$ means failure and 0 no failure. Then the probability of failure depending on t is $p(t) = P(h(X_t) = 1) = \int_D h(x) f_t(x)\ \mathrm{d}x$ and the upper probability of failure $\bar{p}$ is obtained by solving the global optimization problem $\bar{p} = \max_{t\in\mathcal{T}} p(t)$. To explain the situation we start with a numerical example.

2. Numerical Example

We consider the re-calculation of the reliability of a part of an airplane. The corresponding FE-model has $4.7 \cdot 10^6$ degrees of freedom and needs 48 different input parameters. Only the two most decisive input parameters $x = (x_1, x_2)$ are taken into account which are the Young's moduli of two metal components. The output of the FE-computations is the value of the above indicator function h in some complementary components.

The uncertainty about the values x_1 and x_2 is modelled by a family of two-dimensional random variables $\{X_t\}_{t\in\mathcal{T}}$ with $X_t \sim \mathcal{N}(\mu(t), \Sigma(t))$ parametrized by $t = (\varphi, \rho)$. The means $\mu(t)$ may vary on a disc of radius $r = 2$ GPa and midpoint (m, m) with $m = 70$ GPa. Since for upper probabilities the boundary of the disc is relevant we consider only the circle

$$\mu(t) = \mu(\varphi, \rho) = 2 \cdot \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix} + \begin{bmatrix} m \\ m \end{bmatrix},\ \varphi \in [0, 2\pi].$$

The coefficient of correlation varies in an interval, $t_2 = \rho \in [0, 0.8]$, and the coefficient of variation $\nu = 6.5\%$ is assumed to be deterministic. This leads to a parametrization of the covariance matrix

$$\Sigma(t) = \Sigma(\varphi, \rho) = \begin{bmatrix} \sigma_1(\varphi)^2 & \sigma_1(\varphi)\sigma_2(\varphi)\rho \\ \sigma_1(\varphi)\sigma_2(\varphi)\rho & \sigma_2(\varphi)^2 \end{bmatrix}$$

with $\sigma_i(\varphi) = \nu\mu_i(\varphi)$.

3. Estimating $p(t)$ and Their Derivatives

An optimization algorithm for obtaining upper probabilities needs a sequence of parameter values $t_1, \ldots, t_m$ and their probabilities (function values) $p(t_1), \ldots, p(t_m)$. To estimate these probabilities $p(t_i)$ by Monte Carlo simulations it is crucial that for all t_k the simulations are based on the same single set $\omega = \{\omega_1, \ldots, \omega_N\}$ of N random numbers, see Troffaes et al. (2018). These random numbers are then transformed to sets $R_{\omega,t_k} = \{r_1, \ldots, r_N\}$ of sample points distributed according to densities f_{t_k} as for the estimates needed. For parametrized Gaussian distributions it means starting from random numbers $\omega_i \sim \mathcal{N}(0, I)$ and transforming to sample points

$r_i(t) \sim \mathcal{N}(\mu(t), \Sigma(t))$ by $r_i(t) = \mu(t) + C(t)\omega_k$ with Cholesky factor $C(t)$ of $\Sigma(t)$.

Estimating $p(t_k) \approx \frac{1}{|R_{\omega,t_k}|}\sum_{r\in R_{\omega,t_k}} h(r)$ by Monte Carlo simulation independently for all $k = 1,\ldots,m$ would lead to mN evaluations of function h. For reducing this high computational effort it is important for an estimate of $p(t_n)$ in step n of optimization to re-use $h(r)$ and sample points r in sets R_{ω,t_k} from previous steps $k < n$. We use reweighting or importance sampling as in Fetz (2017) and Owen (2018), but here on each set of a partition $D = D_1^n \cap \ldots \cap D_n^n$:

$$p(t_n) = \sum_{k=1}^{n} \int_{D_k^n} h(x)\frac{f_{t_n}(x)}{f_{t_k}(x)} f_{t_k}(x)\,\mathrm{d}x$$
$$\approx \sum_{k=1}^{n} \frac{1}{|R_{\omega,t_k}^n|} \sum_{r\in R_{\omega,t_k}^n} h(r)\frac{f_{t_n}(r)}{f_{t_k}(r)} =: p_{\mathcal{R}_\omega^n}(t_n)$$

where $R_{\omega,t_k}^n = R_{\omega,t_k} \cap D_k^n$ is the set R_{ω,t_k} of sample points restricted to the set D_k^n of the current partition and $\mathcal{R}_\omega^n$ the set $\{R_{\omega,t_1}^n, \ldots, R_{\omega,t_n}^n\}$ of all samples considered in step n. The expensive $h(r)$ is evaluated for $r \in R_{\omega,t_n}^n$ only, all other $h(r)$ are already known from the steps before. The ratios between the original density f_{t_n} and the densities f_{t_k} used instead are the importance sampling ratios which should not be too large (Owen (2018)), because large ratios mean fewer sample points in a considered area leading to less reliable estimates. Taking this into account we define

$$D_k^n = \{x \in D : f_{t_k}(x) \geq \max_{j\leq n, j\neq k} f_{t_j}(x)\}.$$

We do not have to know D_k^n in detail. It is sufficient to know the sets R_{ω,t_k}^n of sample points r keeping only these r where $f_{t_k}(r) > f_{t_j}(r)$, $j = 1,\ldots,n$, $j \neq k$. We generate such sets recursively for $n > 1$ setting $R_{\omega,t_1}^1 = R_{\omega,t_1}$ and

$$R_{\omega,t_n}^n = \{r \in R_{\omega,t_n} : f_{t_n}(r) > \max_{k<n} c_k^{n-1} f_{t_k}(r)\}$$

for the newest and updating all previous ($k < n$):

$$R_{\omega,t_k}^n = \{r \in R_{\omega,t_k}^{n-1} : c_k^{n-1} f_{t_k}(r) > f_{t_n}(r)\}.$$

In addition we may use factors c_k^{n-1} to be more tolerant allowing lower densities to get fewer new sample points. These factors are also recursively defined as $c_k^n = c_n c_k^{n-1}$ where the cumulative multiplications ensure that $R_{\omega,t_k}^n \subseteq \cdots \subseteq R_{\omega,t_k}^{k-1}$. Further we may go one step back in the recursion if $|R_{\omega,t_n}^n|$ is less than some percentage q of N.

For the computation of the i^{th} partial derivative at t_n needed in the optimization algorithm we use $(p_{\mathcal{R}_\omega^n}(t_n + h^{(i)}) - p_{\mathcal{R}_\omega^n}(t_n))/h_i^{(i)}$, $h_j^{(i)} = 10^{-8}\,\delta_{ij}$.

4. Optimizing and Results

For our optimization problem we choose $q = 10\%$, a sequence $c_n = (c_{n-1} - 1)/2 + 1$ with $c_1 = 1.5$ preventing too large tolerances, and $N = 50\,000$. The MATLAB global optimization algorithm needs 2 777 parameter values t_k requiring 218 160 evaluations of h compared to 2 777 N without this new method which re-uses the $h(r)$ from former samples together with good importance sampling ratios on the sets D_k^n of the partition increasing the accuracy of the estimates. In Fig. 1 we show the contour lines of estimates for p (classical and with $p_{\mathcal{R}_\omega^n}$). All parameters t_k used in optimization are visualized as dots where the colour/size indicates the sample sizes used (gray: no new samples, red: new samples needed, the greater the dot the more sample points). We obtain the upper probability $\bar{p} = 0.021$ at $t = (2.438, 0)$.

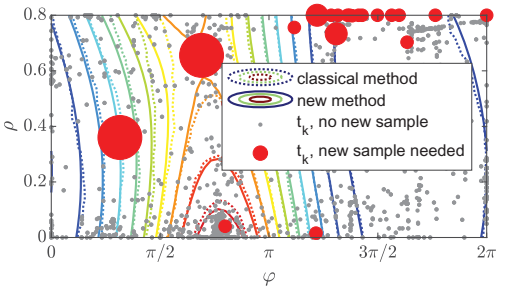

Fig. 1. Contour lines of estimates of p and points t_k.

References

Fetz, T. (2017). Efficient computation of upper probabilities of failure. In C. Bucher, B. R. Ellingwood, and D. M. Frangopol (Eds.), *12th Int. Conference on Structural Safety and Reliability*, pp. 493–502.

Owen, A. B. (2018). Monte Carlo theory, methods and examples. Url: `artowen.su.domains/mc/`.

Troffaes, M. C. M., T. Fetz, and M. Oberguggenberger (2018). Iterative importance sampling for estimating expectation bounds under partial probability specifications. In M. De Angelis (Ed.), *Proc. of the 8th Int. Workshop on Reliable Engineering Computing*, pp. 147–154. Liverpool University Press.