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Optimisation of maintenance by piecewise deterministic Markov processes under conditions of population heterogeneity

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The relevance of a maintenance decision hinges partly on the model's ability to estimate the current health of a system and predict its future evolution based on available information, particularly in cases where access to degradation data is severely limited. This is the context in which we undertake this work. Physics-based approaches can be used to overcome data scarcity, but their models are typically constrained to specific regions. Additionally, degradation phenomena can exhibit highly diverse behaviors that can result in suboptimal maintenance decisions if based solely on population-average degradation performance. Our study aims to explore the potential of Piecewise Deterministic Markov Process (PDMP) in a condition-based maintenance policy when there are variations in behavior within the available sample. We place a strong emphasis on the phenomenon of fatigue cracking. Our demonstration's first phase involves modeling crack evolution behaviors using PDMP-based approaches while identifying the limits of validity of physical models directly from data. We highlight the heterogeneous behaviors of crack evolution. Once we apply a classification algorithm, we define and evaluate a condition-based maintenance strategy tailored to each population.

Keywords: PDMP, Condition based maintenance, Population heterogeneity, Physic based approach, Machine Learning.

1. Introduction

Maximizing the potential of structures heavily relies on the implementation of efficient maintenance strategies, guaranteeing peak performance and extended durability. Recently, there has been a growing focus on Condition-Based Maintenance (CBM). Nonetheless, the effectiveness of the decision-making process in CBM is heavily reliant on the accuracy of the model used to estimate and predict the health of a system. This requires CBM models to incorporate all relevant data in their analyses. Furthermore, these models should be able to adjust to any variations in the operating environment and still provide precise predictions rather than general or approximate estimations. Generating precise predictions can be challenging when there is a lack of data. In such circumstances, physics-based approaches can provide an alternative solution. When constructing (CBM) models for discrete-state deterioration, two methods that have gained widespread adoption are the Markov decision process (MDP) and the semi-Markov decision process (SMDP) Tang et al. (2015); Wang et al. (2008); Chen and Trivedi (2005) . However, these models have a fundamental limitation in that the deterioration process is limited to either a random process with independent increments or a deterministic process. As a result, these models may not be suitable when attempting to describe continuous damage processes, and they are unable to precisely model deterministic processes that incorporate a jump process. Thankfully, the PDMP model Davis (1993) provides a solution to both of these issues. By allowing the process to jump randomly while remaining continuous between jumps, the PDMP model can accurately describe the processes of continuous and deterministic damage. Wang and Chen (2022) have made significant advances in the field of imperfect maintenance modeling by introducing a novel approach that incorporates a piecewise deterministic Markov process. This new model considers both random shocks and natural degradation and incorporates a physical formula to calculate the latter. Similarly, the PDMP model developed by Arismendi et al. (2021) also utilizes distinct degradation states to inform maintenance decisions. Unlike the aforementioned example, the continuous component of the PDMP model's deterministic evolution between two consecutive jumps advances at a constant rate of one, making it a particular case of PDMP known as a piecewise-linear process. In contrast, Lin et al. (2018) framework for modeling and optimizing maintenance of systems focuses on accounting for epistemic uncertainty by describing degradation processes of components using piecewise-deterministic Markov processes. While most maintenance policies discussed above assume uniform degradation patterns across all components, this oversimplification can result in suboptimal decision-making. In practice, components may degrade at varying rates due to different factors. Failing to account for these variations can result in over-maintaining some components while under-maintaining others. One way to achieve this is through on-line monitoring. By analyzing the collected data, it is possible to differentiate components that degrade at different rates , allowing for timely maintenance and replacement. As far as our research has found, there is currently no literature that investigates the use of PDMP to construct a maintenance model that considers the highly heterogeneous behavior of degradation phenomena. The purpose of this paper is to explore the potential of PDMP in the context of maintenance decision, and especially for condition-based maintenance with little data. As mentioned above, the behavior of a degradation of the same system can present several phases that are difficult to capture within the same mathematical model. These phases can be related to the physical phenomena of the degradation or to different operational solicitations that are not always directly observable. We propose in this paper a decision structure that integrates this volatility of the degradation model in a dynamic way. The paper is structured as follows: Section 2 provides a brief overview of the degradation modeling process using a PDMP. Section 3 explains the decision-making process in maintenance and details the cost function used. Finally, Section 4 presents the results obtained from our policy and compares them with existing approaches to evaluate the effectiveness of the proposed maintenance policy.

2. Degradation modeling using Piecewise-deterministic Markov processes

The study of fatigue crack propagation is an intriguing yet demanding topic due to the unpredictability it presents Virkler et al. (1979). While this phenomenon has been extensively studied in the scientific community, the development of physical models that accurately describe the process of crack propagation has become a challenging task. One of the reasons for this challenge is that some physical models have become too parameterized, leading to long calculation times and diminished practicality. Researchers have added numerous parameters to the model in an attempt to capture the complexity of real-world phenomena and respond to specific issues. However, this approach presents another layer of complexity to the modeling process, as these parameters must be estimated from indicators that are correlated to them. The issue with a highly parameterized model is that it becomes difficult to use in practical applications. The model's accuracy is limited by the accuracy of the parameter estimates, and estimating a large number of parameters can be time-consuming and expensive. Additionally, as the number of parameters increases, the model's performance may deteriorate, leading to overfitting and poor generalization to new data. To strike a balance between the complexity of the model and its practical usability, it's important to consider alternative approaches that capture only the essential features of a given phenomenon without overwhelming the model with unnecessary complexity. In fact, utilizing piecewise-deterministic Markov processes is precisely how we plan to proceed Davis (1993). PDMPs are a powerful tool for modeling systems with jumps or discontinuities , as they capture the underlying dynamics of such systems. By using a continuous-time differential equation to model the deterministic component of the dynamics and a jump process to model stochastic jumps, PDMPs provide a more accurate representation of the system's behavior without making the modeling process overly complex. In addition to their accuracy, PDMPs are also highly flexible. They can be constructed using a wide range of differential equations and jump processes, making it easy to tailor them to the specific characteristics of the system being modeled. However, it's worth noting that the physical model used by our PDMP is not universal, as the behavior of the crack can vary depending on its region. To ensure accuracy, the physical model is defined within a specific domain of validity that represents the range of conditions where the model can accurately predict the behavior of the crack. For our analysis, we specifically investigate the Paris-Erdogan model Paris and Erdogan (1963) as our chosen physical model within the scope of this study. This model is well-suited to our needs and provides a solid foundation for our investigation into the behavior of the crack . It is expressed mathematically in the following formula:

$$\frac{da}{dN} = C\left(\Delta\sigma\sqrt{\pi a}\cos\left(\frac{\pi a}{w}\right)\right)^{m} \tag{1}$$

where $\Delta\sigma$ is the stress amplitude, w is the width of the specimen and C and m are material constants. To accurately model the propagation of a fatigue crack through a PDMP denoted as$(a_N, (m, C)_N)_{N\geq 0}$, and account for variability in fatigue testing Virkler et al. (1979), we employ a stochastic modeling approach by randomizing the constant C and m . This approach enables us to capture both the length of the crack a_N and the mode of propagation $(m, C)_N$. As a result, the evolution of the crack in the first regime of propagation is described using the deterministic Paris equation, which involves a couple of parameters (m_1, C_1). An initial condition of $a_0 = 9\,\mathrm{mm}$ is used, and the equation is applied until a random jump time T_S . At this point, the parameters (m_1, C_1) are randomly modified to (m_2, C_2), indicating the start of the second regime of propagation.

3. Maintenance optimization under uncertainty

3.1. *Maintenance decision process*

Our maintenance decision process is structured around six finely-tuned steps.

Step 1: Unveiling Crack Classification through Machine Learning

In this step, our main goal is to precisely determine the class of a sustained crack which is dependent on two crucial assumptions. First, we rely on having access to q measurements taken at an early stage of propagation. Second, we assume that the monitoring metrics we use provide precise representations of the crack's actual size. Since this problem is complex, it requires a sophisticated approach for effective resolution. To tackle this challenge, we employ a data-driven methodology that utilizes supervised Machine Learning techniques to develop a model capable of accurately recognizing the class of a given crack with only a few pieces of information.

Step 2: Calculating the jump time

Once we identify the crack population, we search for the date that can correspond to the jump time T_s using the available q data. In section 4, we delve into the technique utilized for carrying out this task, which is based on the PDMP model.

Step 3: Predicting crack growth

Upon calculating the jump time T_s, we move to predict the propagation of the crack. Although Bruno Sudret Sudret (2007) proposed a Bayesian framework that incorporates early-stage crack measurements to update the joint probability density function of the Paris law parameters and enables prediction of the remaining part of the curve, the experimental curve eventually deviates from the predictions after 175,000 cycles. To unlock even greater accuracy and efficiency in our results, we propose a revamped Metropolis-Hastings algorithm that utilizes techniques from our PDMP modeling, offering a bold new approach.

Step 4: Determining Optimal Decision Variables

The pivotal stage of this study is the determination of the optimal decision variables for our maintenance policy, x_p and τ, based on the observed population. To accomplish this, we employ an optimization algorithm known as the genetic algorithm.

Step 5: Maintenance decision-making

This step involves using the information collected from inspections performed on a specific date τ and associated with a cost c_i to guide the maintenance tasks. Underlying the policy are two key premises. if a crack length reported by an inspection reaches a predefined threshold for preventive maintenance $(x_p \leq a_\tau < x_c)$, the component is replaced to prevent further damage, incurring a preventive maintenance cost c_p. Alternatively, if the component is detected to have failed $(x_c \leq a_\tau)$, corrective maintenance is performed to restore it to its initial state, incurring a corrective maintenance cost c_c. Inspection tasks are assumed to be of negligible duration, and all maintenance actions are expected to be performed instantaneously. If a replacement (whether preventive or corrective) is carried out during the inspection, the maintenance decision process concludes. Otherwise, the process moves on to step 6.

Step 6: Integrating Real-Time Data into Maintenance Decision-Making

To develop more accurate maintenance plans, we integrate real-time inspection data into the maintenance decision-making process. This data is analyzed to identify patterns and trends, enabling the development of new decision rules that are adjusted accordingly.

The maintenance process described above is illustrated in the diagram shown in Figure 1.

3.2. *Economic performance criteria*

To guarantee peak system performance, it is crucial to carefully choose decision parameters. The optimization of this maintenance policy is done using the long-run average cost rate as a criterion

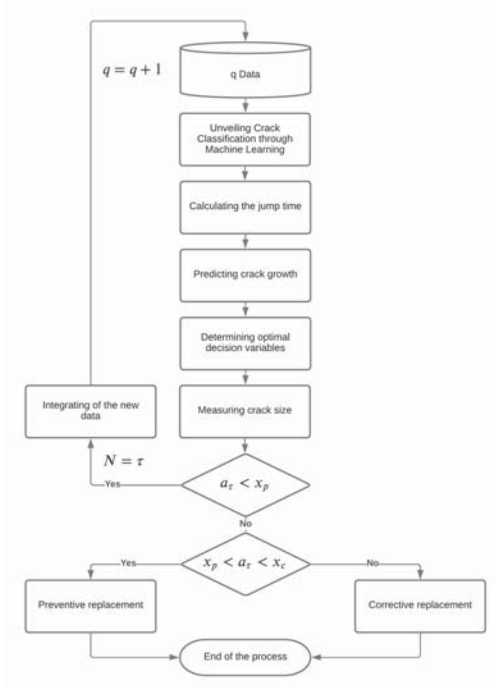

Fig. 1. Strategy diagram.

, which takes into account the expected expenses associated with both preventive and corrective replacements over the lifespan of the crack. Since the aim is to develop a maintenance policy that is both cost-effective and reliable in preventing and correcting potential issues with the system, the problem of maintenance optimization can be defined as:

$$C_\infty(x_p, \tau) = \lim_{N \to +\infty} \frac{C(N)}{N} \tag{2}$$

Where $C(N)$ represents the cumulative maintenance cost at time N . Optimizing this policy amounts to searching for the values of x_p^{opt} and τ^{opt} that minimize the function:

$$C_\infty\left(x_p^{opt}, \tau^{opt}\right) = \min_{x_p,\tau} \left\{C_\infty(x_p, \tau), \tau > 0, 0 < x_p < x_c\right\} \tag{3}$$

In order to evaluate this function, it's necessary to understand how the system behaves when subjected to this policy in steady state. However, because the system is as good as new after each replacement, its evolution restarts independently of its past, resulting in an identical probabilistic behavior each time. Therefore, the evolution of the maintained system is considered a regenerative process, and the points of system replacement are called renewal points. By applying the renewal theorem, it's possible to evaluate the average asymptotic cost, which is the ratio of the average cost over a renewal cycle, $E(C(S))$, to the average length of the cycle, $E(S)$. The equation previously mentioned as (2) can be rewritten as:

$$C_{\infty}(x_p, \tau) = \lim_{N \to +\infty} \frac{C(N)}{N} = \frac{E(C(S))}{E(S)} \quad (4)$$

The following formula calculates the average cost incurred during a renewal cycle :

$$\begin{aligned} &E(C(S)) \\ &= \sum_{k=1}^{+\infty} k \cdot c_i \cdot \left(P\left(x_p \leq a_{k.\tau} < x_c\right) \cdot 1_{\{x_p \leq a_{k.\tau} < x_c\}} \right. \\ &\quad \left. + P\left(x_c \leq a_{k.\tau}\right) \cdot 1_{\{x_c \leq a_{k.\tau}\}} \right) \\ &\quad + c_c \cdot P\left(x_c \leq a_{k.\tau}\right) \cdot 1_{\{x_c \leq a_{k.\tau}\}} \\ &\quad + c_p \cdot P\left(x_p \leq a_{k.\tau} < x_c\right) \cdot 1_{\{x_p \leq a_{k.\tau} < x_c\}} \end{aligned} \quad (5)$$

The equation below is used to calculate the renewal cycle .

$$\begin{aligned} &E(S) \\ &= \sum_{k=1}^{+\infty} k \cdot \tau \cdot \left(P\left(x_p \leq a_{k\cdot\tau} < x_c\right) \cdot 1_{\{x_p \leq a_{k\cdot\tau} < x_c\}} \right. \\ &\quad \left. + P\left(x_c \leq a_{k\cdot\tau}\right) \cdot 1_{\{x_c \leq a_{k\cdot\tau}\}} \right) \end{aligned} \quad (6)$$

Here, k refers to the number of inspections performed, $1_{\{.\}}$ corresponds to the indicator function, which takes on a value of 1 if the argument is true, and a value of 0 otherwise , and $P\left(x_p \leq a_{k.\tau} < x_c\right)$ and $P\left(x_c \leq a_{k\cdot\tau}\right)$ respectively represent the probabilities of preventive and corrective replacement. modeling this maintenance policy can be complex due to the limited applicability of the Paris model to certain regions, particularly given the abundance of possible renewal scenarios. The calculations made are explained in detail in the annex of this document.

4. Results assessment

4.1. *Estimation of the PDMP model parameters*

In this section, we showcase the outcomes of our maintenance policy as it was applied to the Virkler database Virkler et al. (1979). Incorporating this one, we determine the parameters of the PDMP model proposed in Section 2. For each empirical crack, we aim to identify the realizations of the random variables $(m_1, C_1, T_s, m_2, C_2)$ that best align the experimental curve with the theoretical curve defined by these parameters. Our approach to addressing this problem involves formulating an optimization framework as follows:

$$\begin{aligned} &\min_{(m_1, C_1, T_s, m_2, C_2)} f\left(m_1, C_1, T_s, m_2, C_2\right) = \\ &\sum_{i=1}^{\mathrm{q}=164} \left[a_{\text{theo}}^i \left(m_1, C_1, T_s, m_2, C_2\right) - a_{\text{exp}}^i \right]^2 \end{aligned} \quad (7)$$

At each measurement i , we define a_{theo}^i as the theoretical crack length and a_{exp}^i as the experimental crack length. Our PDMP model seems to accurately capture the behavior of the empirical data, as evidenced by the satisfactory fit observed between the simulated and Virkler cracks. In agreement with previous literature on single-regime propagation Perrin (2008), we observe a correlation between m and $\log(C)$ in both regimes of our model. Upon analyzing the heterogeneous degradation behavior present in the Virkler database, we arrived at a decision to categorize it into two distinct classes. The first class includes the eight cracks located at the end of the Virkler bundle, which represent the "slow cracks" category. The second class comprises the remaining cracks within the Virkler bundle, which belong to the "fast cracks" category. As we develop our PDMP model, we rely on estimated parameter statistics that capture the unique characteristics of cracks within each population. Table 1 provides a summary of these results.

4.2. *Example of policy optimization and performance analysis*

To assess the effectiveness of our maintenance policy, we employed Monte Carlo method to simulate the propagation of two population cracks,

Table 1. Statistics of the parameters estimated from the 68 empirical cracks.

	Population 1	Population 2
m_1	$\log \mathcal{N}(0.887, 0.041)$	$\log \mathcal{N}(0.931, 0.069)$
m_2	$\mathcal{N}(2.620, 0.147)$	$\mathcal{N}(2.626, 0.029)$
T_s	$\mathcal{E}(154589)$	$\mathcal{E}(116537)$
$\log(C_1) = h_1(m_1)$	$-5.824m_1 - 9.724$	$-5.565m_1 - 10.244$
$\log(C_2) = h_2(m_2)$	$-6.102m_2 - 8.631$	$-6.065m_2 - 8.667$

using statistical data from Table1. We assumed a limited amount of initial information ($q = 5$) regarding their propagation. Leveraging our training model, we accurately classified these cracks and determined the optimal decision parameters and maintenance cost rate based on intervention costs and failure thresholds ($c_i = 5$, $c_p = 16$, $c_c = 20$, $x_c = 40$), which are presented in Table 2. After applying these decision rules, we determined that both cracks were below their respective preventive threshold. We then integrated the new observations with the initial ones to update our predictions and identify new optimal values. The results are presented in Table 3. As was the case during the first inspection, no replacements were made during the second one. Instead, we established new decision variables, which are presented in Table 4. Based on these variables, we decided to proactively replace both cracks as a preventive measure. The results of our study showed that our proposed maintenance policy recommends carrying out inspections on both cracks at almost the same date, regardless of their population. This indicates that the decision variable determining the inspection date is independent of the population of a given crack. Moreover, when examining the preventive threshold at the first and second inspections, we found that it was nearly identical for both cracks. By studying the Virkler bundle on the dates of these two inspections, it is evident that all cracks behave and deteriorate in the same way, as their density is high. As a result, the preventive threshold is almost identical for these two cracks. However, during the third inspection, our maintenance policy suggested a high failure threshold for a slow crack compared to a fast crack. This approach is logical because, by studying the Virkler bundle on the date of the third inspection, we can see that the different cracks began to behave heterogeneously. This indicates that our maintenance policy is more cautious towards a fast crack than a slow crack. If we shift our attention to the evolution of the long-term average maintenance cost rate, we can observe that integrating new data to our decision process has led to a reduction in cost for both of our case studies. Hence, we can conclude that incorporating real-time monitoring data into our process lowers maintenance costs.

Table 2. The first inspection optimal results.

	τ	x_p	C_∞
Population 1	$9.618e + 04$	33.450	$8.699e - 04$
Population 2	$9.617e + 04$	34.018	$9.159e - 05$

Table 3. The second inspection optimal results.

	τ	x_p	C_∞
Population 1	$1.958e + 05$	34.404	$6.218e - 05$
Population 2	$1.931e + 05$	32.336	$6.772e - 05$

Table 4. The third inspection optimal results.

	τ	x_p	C_∞
Population 1	$2.000e + 05$	29.010	$6.110e - 05$
Population 2	$2.208e + 05$	15.743	$6.178e - 05$

To evaluate the efficacy of our policy, we conducted a comparative analysis with an alternative policy that relied on a physical degradation path model for crack propagation, assumed a homogeneous population, and lacked integration of online data monitoring into the maintenance decision-making process. The outcomes can be viewed in Table 5. The decision rule was applied to the two cracks under study, and based on the preven-

Table 5. The optimal decision variable values and corresponding average long-run cost rate of policy 2.

$\Delta\tau$	x_p	C_∞
$9.618e + 04$	34	$1.043e - 04$

tive limit, no replacement was necessary during the first two inspections. However, in the third inspection, the size of the cracks exceeded the preventive limit, indicating the need for corrective replacement. Therefore, a replacement was made as per the maintenance plan. This results showcases the value of utilizing a PDMP to model the crack propagation process, underscoring the potential benefits of integrating new monitoring data into maintenance decision-making using a well-constructed decision-making framework. While the identification of the crack population did not lead to a significant temporal adaptation of the policy, it did trigger a conditional adaptation, revealing the approach's adaptability and flexibility. Taken together, these findings highlight the efficacy of PDMPs in crack propagation analysis and decision-making, emphasizing the importance of a dynamic and responsive approach to maintenance optimisation.

5. Conclusion

In this paper, we introduce an effective approach for managing fatigue cracks by developing a conditional maintenance policy based on a PDMP model. Our approach incorporates the well-established Paris Erdogan physical law and integrates literature data to estimate the model parameters, enhancing its reliability and accuracy. Real-time monitoring data is also incorporated, which enables dynamic decision-making and further improves our model's predictive ability. Our results demonstrate that integrating real-time monitoring data into the PDMP model reduces costs, thereby making our method a cost-effective alternative to existing approaches. Although we found that resolving heterogeneity did not significantly impact the adaptation of our policy to the population of the crack studied, our approach is highly adaptable and can be applied to more heterogeneous cases with ease. This highlights the versatility and robustness of our approach, making it a valuable tool for informing effective maintenance strategies.

Appendix A. Probabilities of different maintenance interventions

The probability of preventive replacement can be determined by considering the two propagation regimes.

$$\begin{aligned} &P\left(x_p \leq a_{k.\tau} < x_c\right) \\ &= P\left(\left(x_p \leq a_{k.\tau} < x_c\right) \cap \left(k.\tau < T_s\right)\right) \\ &+ P\left(\left(x_p \leq a_{k.\tau} < x_c\right) \cap \left(T_s \leq k.\tau\right)\right) \end{aligned} \tag{A.1}$$

$$\begin{aligned} &P\left(\left(x_p \leq a_{k.\tau} < x_c\right) \cap \left(k\cdot\tau < T_s\right)\right) \\ &= P\left(\left(x_p \leq a_{k\cdot\tau} < x_c\right) \mid \left(k\cdot\tau < T_s\right)\right)\cdot P\left(k\cdot\tau < T_s\right) \\ &= P\left(\left(x_p \leq a_{k.\tau} < x_c\right) \mid m_1\right)\cdot P\left(k\cdot\tau < T_s\right) \\ &= \left(\left(\int_{m_1^{x_p}}^{m_1^{x_c}} f\left(m_1\right)\mathrm{d}m_1\right)\cdot 1_{\{N_p<N_c<T_s\}} \right. \\ &\quad \left. + \left(\int_{m_1^{x_p}}^{m_1^{a_s}} f\left(m_1\right)\mathrm{d}m_1\right)\cdot 1_{\{T_s\leq N_c\}}\right) \\ &\quad \cdot \int_{k.\tau}^{+\infty} f\left(T_\mathrm{s}\right)\mathrm{d}T_\mathrm{s} \end{aligned} \tag{A.2}$$

$$\begin{aligned} &P\left(\left(x_p \leq a_{k.\tau} < x_c\right) \cap \left(T_s \leq k\cdot\tau\right)\right) \\ &= P\left(\left(x_p \leq a_{k.\tau} < x_c\right) \mid \left(T_s \leq k\cdot\tau\right)\right)\cdot P\left(T_s \leq k\cdot\tau\right) \\ &= P\left(\left(x_p \leq a_{k.\tau} < x_c\right) \mid \left(m_1, m_2\right)\right)\cdot P\left(k\cdot\tau \leq T_s\right) \\ &= \left(\left(\int_{m_1}^{m_1^{a_s}} f\left(m_1\right)\mathrm{d}m_1 \cdot \int_0^{m_2^{x_c}} f\left(m_2\right)\mathrm{d}m_2\right)\right. \\ &\quad \cdot 1_{\{0<N_p<T_s\}} \\ &\left. + \left(\int_0^{+\infty} f\left(m_1\right)\mathrm{d}m_1 \cdot \int_{m_2^{x_p}}^{m_2^{x_c}} f\left(m_2\right)\mathrm{d}m_2\right)\cdot 1_{\{T_s\leq N_p\}}\right) \\ &\quad \cdot \int_0^{k\cdot\tau} f\left(T_\mathrm{s}\right)\mathrm{d}T_\mathrm{s} \end{aligned} \tag{A.3}$$

The probability of not replacing can be calculated similarly to the probability of preventive replacement.

$$\begin{aligned} &P\left(a_{k.\tau} < x_p\right) \\ &= P\left(\left(a_{k.\tau} < x_p\right) \cap \left(k.\tau < T_s\right)\right) \\ &+ P\left(\left(a_{k.\tau} < x_p\right) \cap \left(T_s \leq k.\tau\right)\right) \end{aligned} \tag{A.4}$$

$$
\begin{aligned}
&P\left(\left(a_{k.\tau}<x_p\right) \cap\left(k \cdot \tau<T_s\right)\right) \\
&=P\left(\left(a_{k.\tau}<x_p\right) \mid\left(k \cdot \tau<T_s\right)\right) \cdot P\left(k \cdot \tau<T_s\right) \\
&=P\left(\left(a_{k.\tau}<x_p\right) \mid m_1\right) \cdot P\left(k\tau<T_s\right) \\
&=\left(\left(\int_0^{m_1^{x_p}} f\left(m_1\right) \mathrm{d} m_1\right) \cdot 1_{\left\{0<N_p<T_s\right\}}\right. \\
&\left.+\left(\int_0^{m_1^{a_s}} f\left(m_1\right) \mathrm{d} m_1\right) \cdot 1_{\left\{T_s \leq N_p\right\}}\right) \\
&\cdot \int_{k.\tau}^{+\infty} f\left(T_{\mathrm{s}}\right) \mathrm{d} T_{\mathrm{s}}
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
&P\left(\left(a_{k.\tau}<x_p\right) \cap\left(T_s \leq k \cdot \tau\right)\right) \\
&=P\left(\left(a_{k.\tau}<x_p\right) \mid\left(T_s \leq k \cdot \tau\right)\right) \cdot P\left(T_s \leq k \cdot \tau\right) \\
&=P\left(\left(a_{k.\tau}<x_p\right) \mid\left(m_1, m_2\right)\right) \cdot P\left(T_s \leq k \cdot \tau\right) \\
&=\left(\int_0^{+\infty} f\left(m_1\right) \mathrm{d} m_1 \cdot \int_0^{m_2^{x_p}} f\left(m_2\right) \mathrm{d} m_2\right) \\
&\cdot \int_0^{k \cdot \tau} f\left(T_s\right) \mathrm{d} T_s
\end{aligned} \tag{A.6}
$$

The probability of performing a corrective replacement can be calculated based on both the probability of preventive replacement and the probability of not performing any replacements.

$$
\begin{aligned}
&P\left(x_c<a_{k.\tau}\right) \\
&=1-\left(P\left(x_p \leq a_{k.\tau}<x_c\right)+P\left(a_{k.\tau}<x_p\right)\right)
\end{aligned} \tag{A.7}
$$

In the above equations, a_s represents the length of the crack at the jump time T_s, while N_p and N_c respectively refer to the dates on which the crack reached the preventive and failure thresholds. To calculate m_i^x (where $i \in \{0,1\}$) , a polynomial regression of the Paris model is performed from the beginning of regime i until the time of reaching x. The resulting slope of the polynomial represents m_i^x.

References

Arismendi, R., A. Barros, and A. Grall (2021). Piecewise deterministic markov process for condition-based maintenance models. *Reliability Engineering & System Safety 212*, 107540.

Chen, D. and K. S. Trivedi (2005). Optimization for condition-based maintenance with semi-markov decision process. *Reliability engineering & system safety 90*(1), 25–29.

Davis, M. H. (1993). *Markov models & optimization*, Volume 49. CRC Press.

Lin, Y.-H., Y.-F. Li, and E. Zio (2018). A framework for modeling and optimizing maintenance in systems considering epistemic uncertainty and degradation dependence based on pdmps. *IEEE Transactions on Industrial Informatics 14*(1), 210–220.

Paris, P. and F. Erdogan (1963). A critical analysis of crack propagation laws.

Perrin, F. (2008). *Experimental data integration in probabilistic models for the lifetime assessment of structures*. Ph. D. thesis, Ph. D. thesis, Université Blaise Pascal, Clermont-Ferrand, France.

Sudret, B. (2007). Uncertainty propagation and sensitivity analysis in mechanical models. *Habilitationa diriger des recherches, Université Blaise Pascal, Clermont-Ferrand, France 147*, 53.

Tang, D., V. Makis, L. Jafari, and J. Yu (2015). Optimal maintenance policy and residual life estimation for a slowly degrading system subject to condition monitoring. *Reliability Engineering & System Safety 134*, 198–207.

Virkler, D. A., B. Hillberry, and P. K. Goel (1979). The statistical nature of fatigue crack propagation.

Wang, J., S. Hou, Y. Su, J. Du, and W. Wang (2008). Markov decision process based multi-agent system applied to aeroengine maintenance policy optimization. In *2008 Fifth International Conference on Fuzzy Systems and Knowledge Discovery*, Volume 3, pp. 401–408.

Wang, W. and X. Chen (2022). Piecewise deterministic markov process for condition-based imperfect maintenance models. *arXiv preprint arXiv:2211.03023*.