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Optimizing ultrasonic inspection regimes of railway rails

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Ultrasonic inspection of railway rails is considered as an important safety barrier. Fatigue cracks develop due to the cyclic loads of the passing trains, and the objective is to detect the cracks before they develop to rail breakages. It is crucial to understand the speed of crack propagation to define the inspection regime. Trains equipped with ultrasonic instruments can run at a speed of approximately 50 km/h. As the train runs suspects are identified with their position along the track. The ultrasonic scans are investigated and critical suspects are recorded for manual follow up. The manual inspection uses a hand-held trolley also with ultrasonic instruments for a more precise classification. In Norway a classification regime consisting of the categories 2b, 2a, 1 and 0 are used. The main strategy is to monitor 2a and 2b defects, whereas 1 defects have to be fixed within a month, and 0 defects have to be fixed immediately. The inspection- and follow-up regime is now under revision. Markov modelling is the basis for this study. There are several challenges with the Markov model. First of all, since we assume that fatigue is the main failure mechanism, it is not realistic to assume that transition times follow the exponential distribution. Secondly, times between running the inspection car are almost deterministic which requires a special treatment when solving the Chapman-Kolmogorov differential equations. Finally, the follow-up activities are also deterministic, and phase-type models may be used to handle transitions representing the results of the follow-up activities. In this paper we investigate how we can simplify the modelling without compromising the results too much. Statistics from the Norwegian rail network is used for estimating the transition rates.

Keywords: Markov, Maintenance, Ultrasonic Inspection, Rail breakages

1. Introduction

1.1. *Background*

Rail breakages are a serious threat to railway safety, and periodic ultrasonic inspection of the rails is required safety barrier for safe operation of the railway infrastructure. The objective of inspection of the rails are to detect presence of defects such as cracks and track misalignments.

Defects are initiated within the rail and, as the rail operation proceeds, they worsen if no recovery action is undertaken. An ultrasonic inspection car is used to detect potential defects. Candidate defects are verified by a manual inspection with a hand-held trolley with ultrasonic inspection equipment. Defects, or cracks, are assigned a "severity class" and a corresponding maintenance procedure is currently undertaken:

- 2b Keep rail under observation, and perform a new inspection every 3 MBT
- 2a Keep rail under observation, and perform a new inspection every 1 MBT
- 1 Repair the defect quickly, i.e., within one month
- 0 Repair failure immediately and initiate traffic restrictions until failure is fixed.

where MBT is million gross tonnage passed at the specific location. A zero-defect is considered to be a failure, and would develop to a rail breakage in short time, i.e., a state F. From an optimization point of view, both the frequency of inspection and the follow up regime should be optimized.

A common conceptual model used in maintenance is the so-called P-F interval model. The basic idea in our context is that failure is regarded as a two- stage process. First, at some time a defect in the system becomes detectable, i.e., a potential failure (P), then, after some delay-time, the system fails due to the degeneration of the defect, i.e., a failure F. Backer and Christer (1994) present an exhaustive review of the models based on the the P-F interval concept.

The traditional P-F interval model is efficient to determine the frequency of ultrasonic inspection by the measurement car. However, this model is not suited for determining follow-up regimes for the defects. In this work we will align our model with the categories 2b, 2a, 1, 0 and F used in the Norwegian reporting system. It is assumed that all defects follow a trajectory 2b-2a-1-0-F, although since inspections are not continuous, we might expect that some states are not observed. Since fatigue is the dominating failure mechanism, it is not very realistic to claim exponentially distributed sojourn times for each state. In other work, see e.g., Laskowska and Vatn (2020) and Vatn (2020) phase type models for the sojourn time is proposed where virtual sub-states are introduced for each of the main states. In order to keep the model simple, we stick to the assumption that sojourn are exponentially distributed in our basis model, and then we investigate the impact of non-exponential transitions.

1.2. *Objective*

The first objective of this paper is to develop a model giving decision support wrt:

- The frequency of inspection car measurements
- The follow up activities for identified defects, i.e., 2b and 2a defects
- The time limit for fixing type 1 defects
- The time limit for fixing type 2a defects, if this is introduced as a new rule.

The second objective is to propose a method for estimating relevant model parameters based on observed data.

2. Modelling framework

2.1. *Transition diagram*

In the model we introduce two type of states, i.e., hidden states (H) and evident states (E). A hidden state means that there is a defect in the rail, but this is completely unknown for the line manager. An evident state means that the defect has been revealed by ultrasonic inspection. In principle defects could also be detected by other means like line inspections, but in the current model this is not included as part of the modelling framework. Although an evident defect is known to the line manager, it might develop to a more critical state. In the diagram x is representing the actual physical state, i.e., 2b,2a,1,0 or F. Further y is representing the last assessment of the state. The notation used is then:

x-H The physical state is x, and the state is completely unknown for the line manager

x-E The state is x and the state (defect) is revealed by the ultrasonic inspection

x-Ey A fault state has been revealed and the last assessment of the state is y, but the state has developed further, i.e., to the physical state x.

State R represents that a defect has been repaired, and the ultimate objective is to ensure that a defect develops to state R rather than to a fault state.

Several parameters are used to describe the transition between states. τ is the inspection interval. τ_1 and τ_0 are time limits for fixing type 1 and type 0 defects respectively. $\tau_{2\mathrm{a_E}}$ is the time limit to fix a type 2a defect, if such a regime is implemented. τ_x is the follow up interval for defect types $x = 2\mathrm{b}$ and $x = 2\mathrm{a}$ respectively. q_x represents the probability that a physical state x will *not* be detected and correctly classified after an inspection by the ultrasonic car. $q_{x\to y}$ is the probability that a follow up activity of a defect of type x when the physical state is y will fail to reveal the true state. Finally, $\lambda_{x\to y}$ is the transition state from the physical state x to the physical state y.

Figure 1 shows the transition diagram. The vertical transitions and right&down transitions indicate physical degradations. A horizontal transition indicates revealing the true physical state.

Contrary to the PF-model, we may now introduce a failure probability that depends on the state. Typically $q_{2\mathrm{b}} < q_{2\mathrm{a}} < q_1 < q_0$.

A dashed transition means that this transition only occurs at an inspection, where the inspection interval is τ. τ_x is the follow-up interval when the last assessment of the state was x. The state R is representing the situation that a defect is fixed.

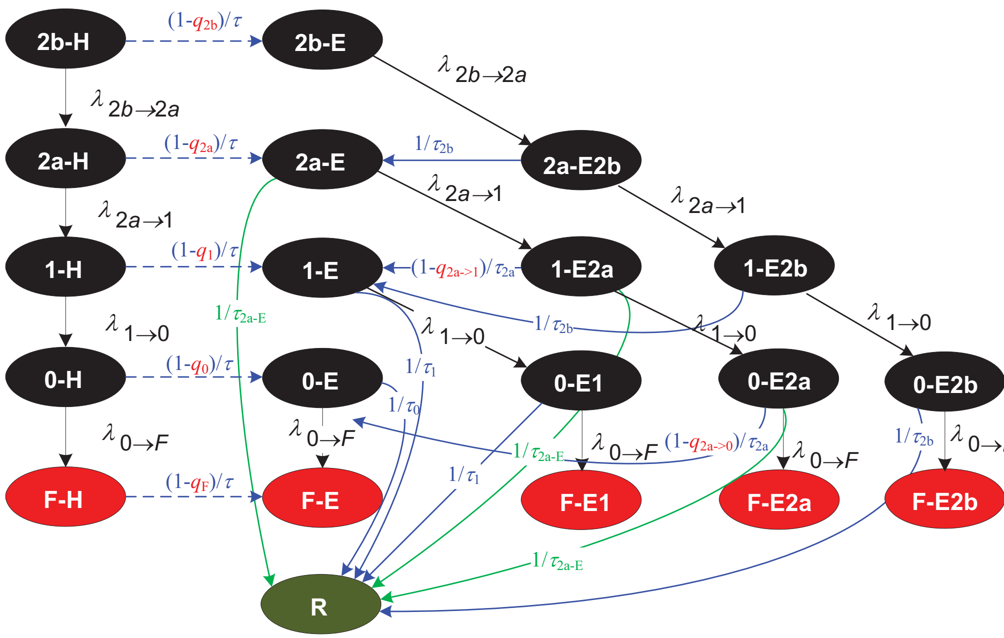

Fig. 1. Markov model. H = hidden, E=Evident, Dashed lines are transitions after inspections.

2.2. *Mathematical modelling*

In Figure 1 there are four types of transitions.

(1) Horizontal- and downwards transitions left to right. These represent physical degradations and are included in a transition matrix $\mathbf{A}$.
(2) Horizontal transitions left to right indicated by dashed arrows. These take place at point of times $\tau, 2\tau, 3\tau, \ldots$, and are not included in $\mathbf{A}$, but in a maintenance matrix $\mathbf{M}$.
(3) Vertical transitions right to left. These transitions are taking place at deterministic point of times given by the follow up intervals. In the model we approximate these with exponential distributed transition times, where phase type modelling later on can improve the approximation. The transitions are included in $\mathbf{A}$.
(4) Transitions from evident states to state R. These transition are restricted to take place within a given time limit, but are approximate by the exponential distribution. These transitions are included in $\mathbf{A}$.

Let $\mathbf{P}(t)$ be the time dependent probability vector for all states in Figure 1. At time $t = 0$ we start in state 2b-H, which means that we do not consider the frequency of defects in this part of the modelling. To find the time dependent probability vector we use the iterative procedure described in Appendix A. I.e., if $\mathbf{P}(t)$ is the solution at time t, then $\mathbf{P}(t + \Delta t)$ is given by

$$\mathbf{P}(t + \Delta t) \approx \mathbf{P}(t) \left[\mathbf{A}\Delta t + \mathbf{I}\right] \qquad (1)$$

where $\mathbf{I}$ is the identity matrix, provided that no inspection by the ultrasonic car takes place between t and $t + \Delta t$. In our case sufficient precision is obtained by letting $\Delta t =$ one day.

To model inspections let $\mathbf{M}$ be the inspection/maintenance matrix. $\mathbf{M}$ is basically the identity matrix, but for a cell with the row representing $x-$H and the column representing $x-$E the value is given by $1 - q_x$, and the corresponding diagonal cell is q_x.

In our integration procedure we apply:

$$\mathbf{P}(t^+) = \mathbf{P}(t^-)\mathbf{M} \quad (2)$$

for $t = \tau, 2\tau, 3\tau, \ldots$.

3. Numerical results

Table 1 shows the parameter values used. The study case is Ofotbanen, an ore freight line in Norway. Figures are preliminary but should be in the correct order of magnitude based on discussion with experts in the field. Statistical data to obtain more precise transition rates are discussed in later sections.

The first four transition rates in Table 1 represent the total journey from a defect is detectable until a failure occurs, i.e., the PF-interval. With the data given this corresponds to an expected PF-interval of 3.4 years with standard deviation 2.4 years. For most lines the PF-interval is assumed to be longer, but since this is an ore line, the PF-interval is assumed to be shorter than the average.

Table 1. Parameter values for base case, time unit is years

Parameter	Value	Description
$\lambda_{2b\to 2a}$	0.5	=1/2
$\lambda_{2a\to 1}$	1	=1/1
$\lambda_{1\to 0}$	1.54	=1/0.65
$\lambda_{0\to F}$	1.43	=1/0.7
q_{2b}	0.25	Pr(Fail to reveal 2b)
q_{2a}	0.2	Pr(Fail to reveal 2a)
q_1	0.1	Pr(Fail to reveal 1)
q_0	0.05	Pr(Fail to reveal 0)
q_F	0.1	Pr(Fail to reveal F)
τ_{2b}	1/2	Follow up interval of 2b
τ_{2a}	2/3	Follow up interval of 2a
τ_1	1/12	Time limit, type 1 repair
τ_0	0.01	Time to repair, type 0
τ	0.5	Interval, measurement car
τ_{2aE}	∞	Potential time limit, 2a

Table 2 shows the result from the sensitivity analysis. The base case corresponds to the input values shown in Table 1. The columns F-H, F-E etc contains the probability that a defect ends up in one of these states. The column F contains the total probability that a defect is not revealed, and ends up as a rail breakage.

For the base case, observe that the state F-E2a has the highest probability. This means that the most likely scenario is that a defect is revealed by the ultrasonic car and classified as a 2a failure, and then it develops unnoticed to a failure. A more strict regime for 2a failures is therefore reasonable.

In the sensitivity analysis we therefore first consider the effect of fixing 2a failures within various time frames. The results shows that compared to the base case we can reduce the total rail breakage probability with 50% by replacing 2a failures within a time frame of half a year. For this scenario we observe that the main contribution to a rail breakage is the state F-E2b, i.e., the defect has been revealed, but it develops unnoticed to a rail breakage.

The sensitivity analysis shows that running the ultrasonic train three times a year gives some reduction in the rail breakage probability, but the value of such a strategy is not any better than setting a relative long time limit for fixing 2a failures, i.e., two years.

More frequent follow up of 2a failures also gives some reduced risk. This is a much cheaper measure than fixing the defect within e.g., one year.

4. Effect of inspection and follow up strategies

In this section we consider two aspects of strategies for handling 2a defects

(1) Frequency, and quality of follow up actions
(2) Lead time for fixing 2a defects

Figure 2 shows the failure probability as a function of lead time if 2a defects are fixed a short period of time after the fault is revealed. Observe that if lead time is long, the failure probability is approaching the value without implementing this strategy.

Figure 3 shows the failure probability as a function of the interval of follow up of 2 defects for three “quality” levels of such a follow up. We see that there is much to gain if we can increase

Table 2. Results for various situations

Situation	F-H	F-E	F-E1	F-E2a	F-E2b	R	Total F
Base case	0.001	0	0.008	0.024	0.014	0.952	0.048
$\tau_{2aE} = 2$	0.001	0	0.005	0.015	0.014	0.964	0.036
$\tau_{2aE} = 1$	0.001	0	0.004	0.01	0.014	0.97	0.03
$\tau_{2aE} = 0.5$	0.001	0	0.004	0.006	0.014	0.975	0.025
$\tau = 1/3$	0	0	0.009	0.02	0.008	0.962	0.038
$\tau = 1$	0.011	0	0.005	0.028	0.027	0.928	0.072
$\tau_{2a} = 0.25$	0.001	0	0.008	0.02	0.014	0.956	0.044

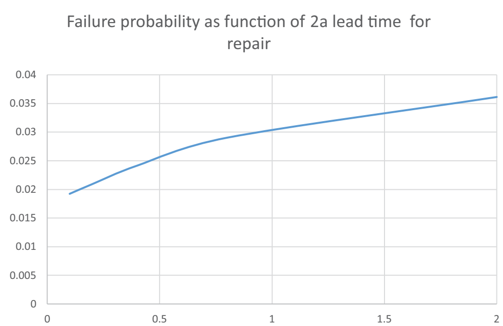

Fig. 2. Effect of various lead time for fixing 2a defects (year)

the probability of revealing a failure. So changing from only visual surface inspection to ultrasonic inspection may pay off.

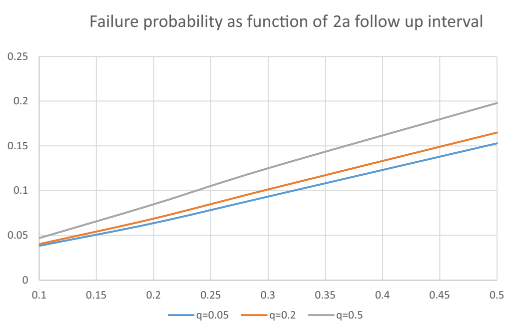

Fig. 3. Effect of various follow up interval of 2a defects (year)

5. Phase type modelling

By means of phase type modelling it is possible to approximate any distribution for the transition times by means of introducing intermediate/artificial states. This means that the states in Figure 1 are replaced by two or more states where we still can work with our Markov framework. The more states we use, the better could the approximation be. See e.g., Laskowska and Vatn (2020) for a description of the approach in a similar setting as used here.

For a complicated diagram the extra states make the model more complicated. We can interpret the set of extra (sub) states as a trajectory out of the main state where we have to run through these states, and typically making the transition more deterministic as we add more sub states. We will demonstrate this for the situation where we implement a strategy to repair 2a defects within a period of τ_{2aE} time units. In reality this would be close to a deterministic transition for which we in our first approach approximated by a exponential distribution. A slightly better approach would be to represent the transition with two exponential transitions by means of one extra state, where the two transition rates are doubled, i.e., $2\tau_{2aE}$. In Figure 1 we see that there are such transitions from the states 2a-E, 1-E2a and 0-E2a. We therefore need to represent these transitions with extra states, say 2a-E*, 1-E2a* and 0-E2a*. Since there is a transition from 2a-E→1-E2a with rate $\lambda_{2a \to 1}$ we also need a similar transition from 2a-E*→1-E2a*, and similarly 1a-E*→0-E2a*.

The model has been re-run with these extra states under the assumption $\tau_{2aE} = 0.5$, and the result shows that the total failure probability is reduced from 2.5% to 2.2%. This is in accordance with Laskowska and Vatn (2020) where more deterministic behaviour typically gives better performance under a given maintenance strategy. Introducing more states will give even more exact results at a cost of a more complicated model. In this study we investigated the effect of introducing phase type models for a very critical parameter, i.e., the time limit for fixing 2a faults, and other parameters could also be considered, but will most likely give less improvement in the model.

6. Estimation of model parameters

6.1. *Introduction*

This section presents ideas for estimation of model parameters, i.e., the degradation rates. In the maintenance system only defects 2b, 2a, 1, 0 and F are used, but in the ultrasonic system also a defect 3 is available, and will be used for the estimation.

Ideally each defect should be documented with a unique identifier and a exact distance measure on the track. This is not the case for the Norwegian system today. This means that defects with uncertain distance measure needs to be paired. In the current study the following criteria applies:

- The distance measure on the track for two subsequent observation should not deviate with more than 20 metres
- Two subsequent observations should be of the same categorization, i.e., a categorization related to the physical mechanism of the defect
- For two subsequent observations, the second one should never be in an earlier stage than the first one.

For each data point $i, i = 1, 2, \ldots, n$ we have the triplet $< u_i, s_{1,i}, s_{2,i} >$, where u_i is the number of days between two subsequent observations, and $s_{1,i}$ and $s_{2,i}$, both $\in$ {3,2b,2a,1,0,F}, are defect types for the first and second observation respectively. It should be noted that in the data defect type F has not been collected so far, so some work remains for a complete estimation.

6.2. *Simple estimation procedure*

A simple estimation procedure is as follows:

(1) Repeat for all defect types $x \in$ {3,2b,2a,1,0}
(2) Set $f = 0$ and $t = 0$
(3) Process all data points $i, i = 1, 2, \ldots, n$
  (a) If $s_{1,i} \neq s_{2,i}$ then let $f = f + 1$
  (b) Let $t = t + u_i / [d(s_{1,i}, s_{2,i} + 1]$, where $d()$ is a distance measure between the first and second observation, i.e., the number of states between the first and second observation. For example $d(2b, 1) = 2$.
(4) The transition rate for defect type, or state x into the next state is $\hat{\lambda}_x = f/t$.

If we assume that transition times out of state x are exponentially distributed, we only need to collect the number of transitions out of state x, i.e., f and the exposure time t. Since a transition out of state x for data point i could have occurred anywhere in the interval of length u_i, we divide by the number of jumps +1, since the exposure time for that observation being in state x is $u_i / [d(s_{1,i}, s_{2,i}) + 1]$. This last argument only holds if all transition rates are equal, but it will do for the simple estimation procedure.

6.3. *The maximum likelihood approach*

A weakness of the simple approach is that it only holds for equal transition rates. Further we do not utilize the information contain in a data point with larger jumps. The The maximum likelihood approach (MLE) approach is rather simple, and it should be noted that, e.g., estimation by utilizing Markov Chain Monte Carlo simulation has been proposed and used by e.g., Bladt and Sørensen (2009) and Laskowska et al. (2023) in a similar situation.

Figure 4 depicts the physical states with the corresponding transition rates. A Markov model is now introduced to represent the physical degradation of the system from state 3 to state F. The transition rates, i.e., the λ's in Figure 4 are above diagonal elements in the transition matrix $\mathbf{A}$, for example $a_{1,2} = \lambda_{3\to2\text{b}}$. The below diagonal elements are vanishing since we do not have data points representing improvements.
The objective of the estimation is to obtain numerical values for $\lambda_{3\to2\text{b}}, \lambda_{2\text{b}\to2\text{a}}, \lambda_{2\text{a}\to1}, \lambda_{1\to0}$ and $\lambda_{0\to\text{F}}$.

The main idea in an MLE approach is to compare the actual transitions taking place between

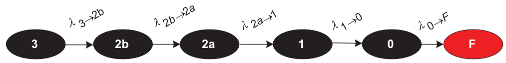

Fig. 4. Model used for estimation

subsequent observations with the probability of that transition should occur, given the parameters in the model, i.e, the λ-vector.

Assume that the system is in state s at time t and we consider a later point of time $t + u$ where no maintenance has been conducted in the period between. Since we know the system state at time t the $\mathbf{P}(t)$-vector is given by

$$P_s(t) = 1$$
$$P_j(t) = 0 \text{ for } j \neq s \tag{3}$$

For a given λ-vector in $\mathbf{A}$ we have:

$$\mathbf{P}(t+u) = \mathbf{P}(t) \cdot e^{\mathbf{A}u} \tag{4}$$

To calculate the exponential of a matrix, .i.e., $e^{\mathbf{A}u}$ is numerically time consuming, and requires efficient library functions. The calculation in Eq. (4) must be carried out for each data point, and for each evaluation of the log-likelihood function, and in Appendix A we therefore propose an efficient approach to speed up the calculation.

To calculate Eq. (4) we let $t = 0$ and let $u = u_i$ for data point i. Further $s_{i,1}$ defines the starting point, i.e., when setting up $\mathbf{P}(0)$. Equation (5) shows the log-likelihood function:

$$l(\lambda_1, \lambda_2, \ldots) = \sum_{j=1} \ln P_{s_{i,2}}(u_i) \tag{5}$$

Table 3 shows the structure of the data needed for the ML estimation:

Table 3. Typical data for ML estimation

i	u_i	$s_{i,1}$	$s_{i,2}$
1	140	2b	2b
2	200	2b	2a
:			

6.4. *Numerical results*

To demonstrate the estimation procedure we use data from another line, i.e., Nordlandsbanen where we have $n = 556$ data points. Table 4 shows summary statistics.

Table 4. Summary statistics

Transition	# data points	Avg u
3→3	146	660
3→2B	8	530
3→2A	116	797
3→1	44	870
3→0	5	552
2B→2B	8	270
2B→2A	13	420
2B→1	9	477
2B→0	10	964
2A→2A	109	483
2A→1	42	589
2A→0	7	264
1→1	30	560
1→0	2	1271
0→0	7	495

The data is rather suspicious since the track is inspected yearly, and the average duration between subsequent observation in the pairs, i.e., a data point typically spans two to three years. There are various reasons for this which is not discussed here.

Table 5 shows the results from the estimation. To better grasp the result the table presents mean time to transition (MTTT) rather than the transition rate.

Table 5. Estimation of mean time to transition

From state	MTTT(simple)	MTTT(MLE)
3	806	878
2B	273	94
2A	1340	1102
1	9037	2059

Note that in the simple approach each state is considered separately. In particular, it should be noted from Table 4 that there are 44 data points indicating an average transition time from state 3 to state 1 equal to 870 hours. This imply evidence for rather high transition rates between state 3 and state 1. Since there are few data points starting in state 2b, the MLE therefore gives a rather high transition rate from state 2b to 2a. In the simple approach, only data for state 2b is used, and hence a lower transition rate and higher MTTT is obtained. Also in the simple approach only data for state 1 is used to calculate the transition rate into state 0, and since there were only two transitions, the 30 data points with no transition drives the transition rate to a low value, and hence a high MTTT, compared to the MLE where there are many transitions into state 0 with rather short transition times, giving a much lover transition rate from state 1 to state 0 compared to the simple approach.

7. Summary and conclusion

This work presents a mathematical framework for modelling degradation of railway rails. The model allows investigating different maintenance strategies. We do not explicitly discuss maintenance optimization, and for a more comprehensive presentation reference is given to previous work documented by Podofillini et al. (2006). We also present an MLE approach for estimating the transition rates in the Markov model used.

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Appendix A. Approximating matrix exponentials

In our MLE approach we need to calculate $\mathbf{P}(t) = \mathbf{P}(0)e^{\mathbf{A}t}$ for each observation each time we have to evaluate the likelihood function. Since calculating the exponential of a matrix is numerically time consuming an approximation is proposed. Recall that $\mathbf{P}(t) = \mathbf{P}(0)e^{\mathbf{A}t}$ was obtained from the Kolmogorov Markov equations:

$$\mathbf{P}(t) \cdot \mathbf{A} = \dot{\mathbf{P}}(t) \quad \text{(A.1)}$$

But rather than solving the exponential, we realize that for Δt being small, we have:

$$\mathbf{P}(t) \cdot \mathbf{A} \approx [\mathbf{P}(t + \Delta t) - \mathbf{P}(t)] / \Delta t \quad \text{(A.2)}$$

which leads to an iterative procedure where we repeatedly use:

$$\mathbf{P}(t + \Delta t) \approx \mathbf{P}(t) [\mathbf{A}\Delta t + \mathbf{I}] \quad \text{(A.3)}$$

and where $\mathbf{I}$ is the identity matrix. Calculating Eq. (A.3) is numerically fast. However, if we need to calculate for e.g., one year, i.e., $t = 365$ this will be time consuming. Now let $\mathbf{M}_0 = [\mathbf{A}\Delta t + \mathbf{I}]$, and calculate subsequently:

$$\mathbf{M}_i = \mathbf{M}_{i-1} \cdot \mathbf{M}_{i-1} \quad \text{(A.4)}$$

as long as $2^{i-1} < t_{\max}$. Now it follows that for $t = 1, 2, 4, 8 \ldots$, we use

$$\mathbf{P}(t) \approx \mathbf{P}(0) \cdot \mathbf{M}_i \quad \text{(A.5)}$$

where $2^{i-1} = t$. If $t \notin \{1, 2, 4, 8\}$ let $\mathbf{b}$ be a vector for the binary representation of t, for example $\mathbf{b} = [0, 1, 0, 1, 0, 0, 0, ...]$ corresponds to $t = 0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 + 0 \cdot 2^4 + \ldots = 0 + 2 + 0 + 8 + 0 \ldots = 10$. In a similar way we calculate $\mathbf{P}(t = 10) = \mathbf{P}(0) \cdot \mathbf{M}_1 \cdot \mathbf{M}_3$.

Note that if the longest observation period is 2048 days, i.e., five and a half year, we only need 11 matrix multiplication to generate all the $\mathbf{M}_i$ matrices. This is done only once for each evaluation of the likelihood function. For each observation we typically need in average five or six matrix multiplications to calculate $\mathbf{P}(t)$.

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