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Consideration of Uncertainty in Reliability Demonstration Test Planning

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The risk of a failed test (type-II statistical error) is rarely considered in both End of Life and Success Run Tests (Zero Failure Tests) in terms of reliability demonstration testing. In order to make the right decision under uncertainty, the remaining risk can be calculated for the considered sample size using numerical or approximative-analytical approaches. As with any other hypothesis test, prior knowledge is necessary to estimate the distribution of the alternative hypothesis. If this prior knowledge is stemming from life tests with very small sample sizes, the information is subjected to uncertainty, that has to be considered when planning the test. This paper presents an approach to consider the uncertainty in reliability demonstration test planning within a numerical-bootstrap and an approximative-analytical approach. The implications are illustrated using some exemplary results.

Keywords: Uncertainty, Probability of Test Success, Reliability Demonstration, Test Planning

1. Introduction

One of the main development requirements for a product is to demonstrate its service life with correspondingly high reliability (Bertsche 2023). Frequent and premature failures, especially in the case of new products, can cause companies not only considerable economic damage, but also damage to their image. Particularly with new and increasingly complex products, reliability is becoming more and more important. Reliability tests are usually used to demonstrate a required reliability. However, there are some challenges in planning these tests, especially with regard to resource efficiency. This is because lifetime tests on a prototype basis are generally always associated with increased expense.

The central metric in the life test planning context is the Probability of Test Success P_{ts}, which can be used to determine the statistical power of the reliability test based on prior knowledge of the product's failure behavior. Its basic applicability has already been demonstrated in (Dazer, 2020), (Grundler 2022). P_{ts} has also already been successfully applied to accelerated life testing (Herzig 2020) and to system and component testing (Grundler 2020). In addition to calculate the necessary number of samples for reliability demonstration, it can also be used to compare the available test strategies. Thus, the best possible test strategy can be selected for individual scenarios.

2. Challenge: Uncertainty in Prior Knowledge

Due to the stochastic lifetime, the planning of life tests always requires prior knowledge about the failure behavior. Therefore, the quality of the prior knowledge is crucial for a good planning result. The prior knowledge can stem from different sources such as early prototype tests, similar applications, expert knowledge or lifetime calculations (Grundler 2019).

Each source of information contains uncertainty. In principle, uncertainty can be divided into an aleatory and an epistemic part. The aleatory uncertainty can be described easily via a Weibull distribution. Even from expert estimates, reasonable assumptions for a Weibull distribution can be made with a lot of product experience. Although there have been some studies for specific applications (Grundler 2021), the consideration of the epistemic uncertainty of the prior knowledge in the planning process has not been successfully implemented so far. Accordingly, this paper presents two approaches to consider the epistemic uncertainty in the reliability test planning process for reliability demonstration. Consideration of prior knowledge uncertainty is essential, especially for sources with high uncertainty, since in such cases the planning process should also be approached much more critically.

3. State of the Art of Calculating P_{ts}

In order to be able to explain the calculation steps for the integration and thus for the consideration of the epistemic uncertainty in the prior knowledge, a basic understanding of the current calculations is necessary. In principle, life tests can be divided into failure-free and failure-based test strategies. For both representative's own calculation approaches for the planning of reliability demonstration tests exist.

3.1. P_{ts} for End-of-Life Tests

For the End-of-life (EoL) test, the calculation of the P_{ts} is based on a general hypothesis testing idea (Grundler 2022). For the non-normally distributed life data, the probability must be calculated for the relevant lifetime quantile (usually derived from the reliability requirement), with which the test can provide reliability demonstration. This probability corresponds to the statistical power in the general context of testing methodology. Dazer, Grundler and Herzig referred to it in the context of reliability demonstration tests as the Probability of Test Success P_{ts} (Dazer 2020, Grundler 2020, Herzig 2020). Fig. 1 shows all the necessary relationships.

The two relevant hypotheses for reliability demonstration using an EoL test are:

$$H_0: \tau < 0 \quad (1)$$

$$H_1: \tau \geq 0 \quad (2)$$

The test statistic τ used here is the difference of the estimated lifetime at required reliability $t_{R_{req}}$ to the required lifetime, hence $\tau = t_{R_{req}} - t_{req}$. To calculate the P_{ts}, the distributions of the null hypothesis f_{H_0} and of the alternative hypothesis f_{H_1} must be calculated. The P_{ts} can then be determined using the integral:

$$P_{ts} = \int_{t_{crit}}^{\infty} f_{H_1}(\tau) d\tau \quad (3)$$

The distributions can be described in a numerical way or in an analytical-approximative way. While t_{crit} is calculated using the required Confidence C and $C = \int_{-\infty}^{t_{crit}} f_{H_0}(\tau) d\tau$.

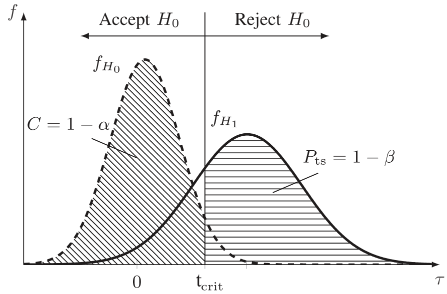

Fig. 1. Relationships of f_{H_0} & f_{H_1} and the resulting integrals of C and P_{ts} (Grundler 2022)

3.1.1. *Numerical Bootstrap Procedure*

To plan EoL tests, the first step of the bootstrap approach is to generate n pseudo-random numbers from the failure distribution $F(t)$ of prior knowledge. The sample size and the pattern (e.g. censoring) according to which these failure times are generated must correspond to the reliability demonstration test that is to be planned. The failure distribution $\hat{F}(t)$ of this bootstrap-sample is estimated using MLE. Afterwards the lifetime at the required reliability $\hat{t}_{R_{req}}$ is calculated using $\hat{F}(t)$. Since the failure distribution of the prior knowledge is linked to the alternative hypothesis H_1, a value of the test statistic under validity of the alternative hypothesis $\hat{\tau}_{H_1}$ can be obtained by subtracting the required lifetime from this calculated lifetime:

$$\begin{aligned} \hat{\tau}_{H_1} &= \hat{t}_{R_{req}} - t_{req} \\ &= \hat{F}^{-1}(1 - R_{req}) - t_{req} \end{aligned} \quad (4)$$

Values of the test statistic under validity of the null hypothesis H_0 can be calculated in a similar way. But, the failure distribution used must correspond to H_0 (reliability requirement). For this purpose, the bootstrap failure times already generated are multiplied by $(1 - s)$ so that the limiting case $s = 0$ is reached. This multiplicative transformation ensures that the shape of the failure distribution $F(t)$ is preserved. Here, s describes the relationship between the requirement and the real product reliability (lifetime safety factor), which is approximated as best as possible by prior knowledge. Therefore, for the transformed distribution, the lifetime quantile for required reliability for that limiting case is $t_p = t_{req}$. The test statistic under validity of H_0 is then obtained as:

$$\hat{\tau}_{H_0} = (1-s)\cdot \hat{t}_{R_{req}} - t_{req} = t_{req}\cdot\left(\frac{\hat{F}^{-1}(1-R_{req})}{F^{-1}(1-R_{req})} - 1\right) \quad (5)$$

Using multiple bootstrap iterations, the distribution of the test statistic under validity of both hypotheses can be described either empirically or parametrically using the generated values of test statistics. A more detailed explanation of the procedure can be found in (Grundler 2022).

3.1.2. *Approximative-analytical approach*

The motivation for the analytical-approximative method is to find a fast-computable solution, because the bootstrap approach can be associated with a very high computation time, especially for a large parameter space. Therefore, the central limit theorem is used to approximate the distributions of τ_{H_0} and τ_{H_1}. Accordingly, for finite sample sizes, the normal distribution can be used as an approximation to the distribution of the test statistic. According to the central limit theorem, the distribution of the sample quantile of a known distribution $F(t)$, i.e. the empirically formed quantile, is normally distributed with the parameters (Fisher 1915):

$$\mu = F^{-1}(q) \quad (6)$$

$$\sigma = \sqrt{\frac{q\cdot(1-q)}{n\cdot f\left(F^{-1}(q)\right)^2}} \quad (7)$$

where the q-quantile of the distribution $F(t)$ is formed from its inverse function $F^{-1}(q)$. $f(t)$ is the density function of $F(t)$ and n is the sample size. Using the test statistic from Eq. 4 and the asymptotic behaviour from Eq. 6 & 7, the approximate distributions of τ_{H_0} and τ_{H_1} can be determined as normal distributions as follows (Grundler 2022):

$$\tau_{H_0} \sim \mathcal{N}\left(0, \sqrt{\frac{R_{req}\cdot(1-R_{req})}{n\cdot f_{H_0}(t_{req})^2}}\right) \quad (8)$$

$$\tau_{H_1} \sim \mathcal{N}\left(t_p - t_{req}, \sqrt{\frac{R_{req}\cdot(1-R_{req})}{n\cdot f_{H_1}(t_{req})^2}}\right) \quad (9)$$

f_{H_0} is the transformed distribution from prior knowledge according to H_0. If prior knowledge is formulated using a Weibull distribution with scale parameter T_{H_1}, one can calculate the transformed scale parameter T_{H_0} as follows:

$$T_{H_0} = (1-s)\cdot T_{H_1} = \frac{t_{req}}{t_p}\cdot T_{H_1} \quad (10)$$

The P_{ts} can then be determined via the approximated normal distribution Φ:

$$P_{ts} = 1 - \Phi\left(\Phi^{-1}\left(C_{req}; 0, \sqrt{\frac{R_{req}\cdot(1-R_{req})}{n\cdot f_{H_0}(t_{req})^2}}\right); t_p - t_{req}, \sqrt{\frac{R_{req}\cdot(1-R_{req})}{n\cdot f_{H_0}(t_p)^2}}\right) \quad (11)$$

Here, the lifetime quantile is determined as an empirical sample quantile.

Due to the multiplicative correlation between the failure times of H_1 and H_0 using s, this relationship is also valid for the corresponding life quantiles and therefore also for the relationship between the alternative distribution and null distribution. For this reason, there is the following relationship of the standard deviations of the alternative and null distribution:

$$\sigma_{H_0} = (1-s)\cdot\sigma_{H_1} = \frac{t_{req}}{T_{H_1}}\cdot\left(-\ln(R_{req})\right)^{-1/b}\cdot\sigma_{H_1} \quad (12)$$

Here, it is assumed that no estimation of the failure distribution takes place, i.e. via an MLE. This also means that censoring can only be considered to a very limited extent. To overcome this additional drawback, the scale and variance of the lifetime quantile can also be calculated using the asymptotic properties of the MLE and the variance-covariance matrix. For the complete derivation, we refer to (Grundler, 2022).

The scale and variance of the test statistic distribution of the alternative hypothesis is obtained using the Weibull distribution (Grundler, 2022):

$$t_{R_{req},H_1} \sim \mathcal{N}\left(T_{H_1}\left(-ln(R_{req})\right)^{1/b_{H_1}}, \sigma_{H_1}\right) \quad (13)$$

with:

$$\sigma_{H_1} = \left(\left(-\ln(R_{req})\right)^{2/b_{H_1}}\cdot Var(T_{H_1}) + \frac{T_{H_1}^2}{b_{H_1}^4}\cdot\ln\left(-\ln(R_{req})\right)^2\cdot\left(-\ln(R_{req})\right)^{2/b}\cdot Var(b_{H_1}) - \frac{2T_{H_1}^2}{b_{H_1}^2}\cdot\ln\left(-ln(R_{req})\right)\cdot\left(-\ln(R_{req})\right)^{2/b}\cdot Cov(T_{H_1}, b_{H_1})\right)^{1/2} \quad (14)$$

T_{H_1} and b_{H_1} are the Weibull scale and shape parameter stemming from prior knowledge considering the assumed failure behavior of H_1. The transformation using s applied in the bootstrap approach can also be used here to calculate the null hypothesis. The P_{ts} then results in:

$$P_{ts} = 1 - \Phi\left(\Phi^{-1}\left(C_{req}; 0, \sigma_{H_0}\right); T_{H_1}\left(-\ln\left(R_{req}\right)\right)^{1/b} - t_{req}, \sigma_{H_1}\right) \quad (15)$$

For accuracy and comparisons of the calculation methods, we refer to (Grundler 2022).

3.2. P_{ts} for Success Run Tests

The Success Run (SR) Test is based on a binary classification in which all specimens are tested up to a predefined lifetime. Each specimen is simply assigned as "failed" or "passed". Therefore, the binomial distribution can be used as a planning approach:

$$C = 1 - \sum_{i=0}^{k} \binom{n}{i} \cdot \left(R_{req}\right)^{n-i} \cdot \left(1 - R_{req}\right)^{i} \quad (16)$$

This corresponds to the null hypothesis for the limiting case of $s = 0$. In analogy to Eq. 16, the P_{ts} of a SR Test can be calculated analytically and exactly using the following binomial distribution:

$$P_{ts} = \sum_{i=0}^{k} \binom{n}{i} \cdot \left(R_p\right)^{n-i} \cdot \left(1 - R_p\right)^{i} \quad (17)$$

Instead of the required reliability, the reliability at the required lifetime t_{req} corresponding to the prior knowledge $R_p\left(t_{req}\right) = 1 - F\left(t_{req}\right)$ is used here as the complement of the success probability parameter p of the binomial distribution.

If the reliability according to prior knowledge is greater than or equal to the requirement $R_p\left(t_{req}\right) \geq R_{req}\left(t_{req}\right)$, this corresponds to the alternative hypothesis. Due to the relationship between the binomial and beta distributions, Eq. 16 & 17 can also be written as beta distributions:

$$C = \int_{R_{req}}^{1} \frac{R^{n-k-1} \cdot (1-R)^{k}}{\beta(n-k, k+1)} dR \quad (18)$$

$$P_{ts} = \int_{0}^{R_p} \frac{R^{n-k-1} \cdot (1-R)^{k}}{\beta(n-k, k+1)} dR \quad (19)$$

It is the same beta distribution $\boldsymbol{\mathcal{B}}(n-k, k+1)$ in both cases, because the resulting reliability distribution is defined solely by the number of passed and failed specimens.

Since Confidence and P_{ts} are calculated from the same distribution they only differ in the integral limits. It is obvious that the P_{ts} becomes the complement of the required Confidence i.e., $P_{ts} \rightarrow C$ when the required reliability approaches the actual reliability, i.e. for $R_p \rightarrow R_{req}$ and $s \rightarrow 0$, respectively. The relationship of P_{ts} and Confidence is shown in Figure 2.

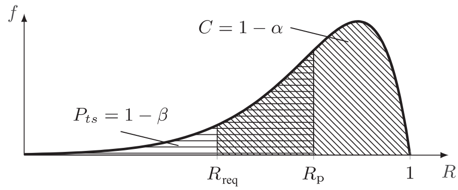

Fig. 2. Beta distribution of the SR test and the corresponding integrals for C and P_{ts} (Grundler 2022)

4. Consideration of Uncertainty in Prior Knowledge in the P_{ts} calculation procedure

The information about failure times is usually coming from one or more observations. Due to the always limited amount of observations, the information is subject to epistemic uncertainty. Thus, the prior knowledge about the reliability and the failure distribution in particular are also subject to uncertainty. In order to be able to consider these in the considerations of the reliability demonstration planning the following methods and procedures are presented and the calculation of the P_{ts} is introduced.

4.2. Uncertainty in Prior Knowledge

The reliability is the aleatoric uncertainty of the product's lifetime. The lifetime is varying from product to product and is therefore described by a probability. The estimation of the actual underlying probability of the lifetime can only be made inadequately, i.e., not exactly due to the sample error. This inadequacy of observation is called epistemic uncertainty.

Prior knowledge in the context of reliability corresponds to information about the reliability itself (type SR test: reliability distribution) or information about the failure distribution (type EoL test: failure distribution or entire sample). Thus, it is the aleatory uncertainty - the uncertainty about the failure times, or reliability. The uncertainty about the correct determination of reliability (epistemic uncertainty) must be additionally considered in the test planning process, because prior knowledge is always subject to epistemic uncertainty.

The two types of prior knowledge in certain (fixed values) and uncertain form are shown in Figure 3. Thus, the epistemic uncertainty in prior knowledge can be determined by specifying the reliability distribution, for example, by a beta distribution for a SR Test. The uncertainty in prior knowledge within an EOL test can be determined, e.g., by the original sample size of the failure distribution (Grundler 2022).

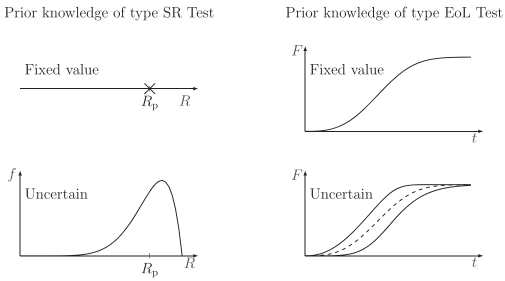

Fig. 3. Types of prior knowledge in its certain (fixed values) and uncertain form

4.2. Calculation of P_{ts} with Uncertainty in Prior Knowledge

In order to calculate the P_{ts}, the epistemic uncertainty of prior knowledge must be captured, i.e., the variance of reliability (prior knowledge SR Test) or the variance of the parameters of the failure distribution (prior knowledge EoL Test).

4.2.1 *Calculation of P_{ts} with Uncertainty for EoL Tests*

Prerequisite for the calculation of P_{ts} in EoL Tests is prior knowledge of the type EoL Test. That is a failure distribution with specification of the original sample size or the specification of the original sample in the form of failure and suspension times itself. The procedure is formulated for the two-parameter Weibull distribution. However, the procedure can also be applied to other distributions. In order to determine the variance of the parameters of the failure distribution, the existing procedure is extended to a double bootstrap approach. Here, in a first step, a bootstrap sample of the size of prior knowledge n_0 is generated from the failure distribution. This can be done parametrically or non-parametrically. This sample is then used to determine the parameters of the failure distribution, which in turn are used again to generate another bootstrap sample of size n and censoring of the test to be planned and evaluated. By this double procedure, both sample sizes (n_0 and n) are considered and the corresponding variance and uncertainty of these combined sample sizes can be captured. Further computation is performed as in the case of certain prior knowledge. The extended flow chart for this procedure is shown in Figure 4. Due to the bootstrap procedure applied twice here, the number of iterations must be chosen higher than in the simple case without uncertainty in prior knowledge.

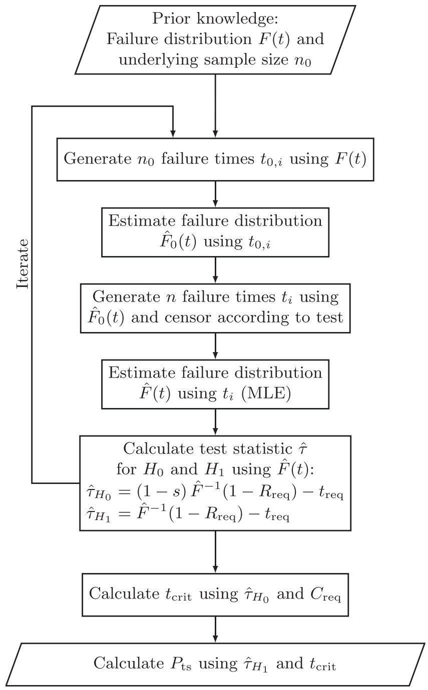

Fig. 4. Flowchart to calculate P_{ts} with consideration of uncertainty in prior knowledge

The methods described in the analytical-approximative approach use the asymptotic distribution of the lifetime quantile to calculate the P_{ts} in an analytical way (Grundler 2022). Here, the asymptotic standard deviations σ_{H_0} and σ_{H_1} are determined under the validity of the two hypotheses using the Central Limit Theorem and a Taylor series approximation. The distributions of the lifetime quantiles are then obtained as normal distributions. The scale parameters μ_{H_0} and μ_{H_1} of these distributions are predetermined via prior knowledge, because they are specified by the defined hypotheses and are independent of the Central Limit Theorem considerations and the sample size. However, because of the uncertainty in prior knowledge, it is not possible to further specify a single value for the parameters of the failure distribution, which means that the scale of the lifetime quantile is also subjected to variance. However, with the same approach, this variance can also be determined as a normal distribution via the Central Limit Theorem and

Taylor series approximation. For this purpose, only the sample size n_0 on which the prior knowledge was based, has to be used. Thus, the normal distribution of the scale of the lifetime quantile under validity of the alternative hypothesis μ_{H_1} results as:

$$\mu_{H_1} \sim \mathcal{N}\left(T_{H_1}\left(-\ln\left(R_{req}\right)\right)^{1/b}, \sigma_{H_1,n_0}\right) \tag{20}$$

However, it is important to note that the variance in this equation is determined by means of the synthetic failure times corresponding to n_0, instead of n. The relationships between the likelihood, its derivatives and the variances and covariances are still valid, see (Grundler 2022).

The scale parameter of the asymptotic distribution of the scale of the lifetime quantile is identical with the scale of the lifetime quantile in case no uncertainty would be considered. Using the transformation of Eq. 10, the normal distribution of the scale of the lifetime quantile for the validity of H_0 results in:

$$\mu_{H_0} \sim \mathcal{N}\left(t_{req}, (1-s)\cdot\sigma_{H_1,n_0}\right) = \mathcal{N}\left(t_{req}, \frac{t_{req}}{T_{H_1}}\left(-\ln\left(R_{req}\right)\right)^{-1/b}\cdot\sigma_{H_1,n_0}\right) \tag{21}$$

with:

$$\sigma_{H_0,n_0} = (1-s)\cdot\sigma_{H_1,n_0} = \frac{t_{req}}{T_{H_1}}\left(-\ln\left(R_{req}\right)\right)^{-1/b}\cdot\sigma_{H_1,n_0} \tag{22}$$

In such a case, when the scale parameter of a normal distribution is again normally distributed, it is also called a mixture distribution and the resulting distribution is a normal distribution again (Gneiting 1997). The scale parameter of the resulting distribution corresponds to that of the distribution which describes the scattering scale parameter. The variances are summed up, which means that the resulting standard deviation σ_Σ is the geometric sum of the standard deviations of the two distributions. Thus, the asymptotic normal distribution of the lifetime quantile under the validity of H_1 with consideration of the uncertainty of the prior knowledge is determined as follows:

$$t_{R_{req},H_1} \sim \mathcal{N}\left(t_p, \sigma_{\Sigma,H_1}\right) = \mathcal{N}\left(T_{H_1}\left(-ln\left(R_{req}\right)\right)^{1/b}, \sqrt{\sigma_{H_1,n_0}^2 + \sigma_{H_1}^2}\right) \tag{23}$$

Accordingly, the following distribution applies analogously to the validity of the null hypothesis:

$$t_{R_{req},H_0} \sim \mathcal{N}\left(t_{req}, \sigma_{\Sigma,H_0}\right) \tag{24}$$

with:

$$\sigma_{\Sigma,H_0} = \sqrt{\sigma_{H_0,n_0}^2 + \sigma_{H_0}^2} = \frac{t_{req}}{T_{H_1}}\left(-\ln\left(R_{req}\right)\right)^{-1/b}\cdot\sqrt{\sigma_{H_1,n_0}^2 + \sigma_{H_1}^2} = (1-s)\cdot\sigma_{\Sigma,H_1} \tag{25}$$

Finally, P_{ts} resulting in:

$$P_{ts} = 1 - \Phi\left(\Phi^{-1}\left(C_{req}; 0, \frac{t_{req}}{T_{H_1}}\left(-\ln\left(R_{req}\right)\right)^{-1/b}, \sigma_{\Sigma,H_1}\right); T_{H_1}\left(-\ln\left(R_{req}\right)\right)^{1/b} - t_{req}, \sigma_{\Sigma,H_1}\right) \tag{26}$$

Based on the equations presented, it can be seen that the P_{ts} of EoL tests while considering the uncertainty in prior knowledge can reach at most the value that the sample size of the prior knowledge is reaching in a calculation without uncertainty. This is due to the summation of the variances. The variance of the lifetime quantile can therefore never be smaller than the variance that results from the sample size of the prior knowledge alone. For practical purposes, this means, that the values of P_{ts} will always be smaller if the uncertainty in the prior knowledge is considered. Furthermore, a sample size larger than that of the prior knowledge $n > n_0$ would not lead to an increase in the P_{ts}.

4.2.2 *Calculation of P_{ts} with Uncertainty for Success Run Tests*

If the uncertainty in the prior knowledge is to be considered for SR Test, the prior knowledge must first be translated into a suitable form if necessary. If the prior knowledge is available in the form of a beta distribution, it can be used directly.

The approach to calculate the P_{ts} from Eq. 17 using the binomial distribution cannot be further used here, because the parameter of the success probability, which in this context is the probability of failure, is not further a single value, but scatters according to the beta distribution of prior knowledge. However, this corresponds exactly to the information given by the beta-binomial distribution. Accordingly, the P_{ts} with prior knowledge can be calculated as a beta distribution as follows to account for the uncertainty in the prior knowledge (Grundler 2022):

$$P_{ts} = \sum_{i=0}^{k}\binom{n}{i}\cdot\frac{\beta(B+i, A+n-i)}{\beta(B,A)} \tag{27}$$

The beta distribution with the parameters A and B is describing the reliability distribution for the required service life. In the planned SR Test with sample size n a maximum of k failures are allowed.

4.3 Influence of Uncertainty in Prior Knowledge

The calculation of the P_{ts} considering the uncertainty in prior knowledge holds potential, because on the one hand one can calculate P_{ts}

more realistically and on the other hand one can assess the quality of the available prior knowledge indirectly.

4.3.1 *Comparison of calculation approaches for EoL tests*

In order to get a general understanding of the two calculation approaches for EoL tests, they are compared for a Weibull distributed failure behaviour with $T_{H_1} = 1$ and $b_{H_1} = 3.5$. For a reliability requirement of $R_{req} = 95$ %, $C = 90$ % and $t_{req} = 0.3$, the values of P_{ts} are shown in Figure 5 with regard to the sample size of the test n als well as the sample size n_0 on which the prior knowledge about the failure distribution is based on.

It is obvious, that the analytical approach results in a symmetric surface to the n and n_0 axes. Which means the two sample sizes have equal impact on the P_{ts} values and the sample size of prior knowledge concerning its uncertainty n_0, is restricting the maximum achievable values of P_{ts}. This is in contrast to the values calculated using the bootstrap approach. Here an increase in sample size of the EoL test n results in an increase in P_{ts}. This is in contradiction with the observations on the superposition of the standard deviations of the lifetime quantile and its location parameter, see Eq. 25. This behavior can be explained by the bias of the MLE estimate with respect to the Weibull shape parameter b (see e.g. (Tevetoglu 2020)). For small sample sizes n_0, the shape parameter is slightly overestimated in the first bootstrap step.

This results in the fact that in the second bootstrap step, when generating the bootstrap sample of size n, the failure times are less scattered than required to correctly represent the uncertainty. The Weibull parameters thereby re-estimated are then overestimated again for small n. Thus, this double bootstrap approach amplifies the overestimation of the shape parameter due to the bias of the MLE. Consequently, the variance of the lifetime quantile is too small under the validity of both hypotheses, which in turn results in increased values of P_{ts}. In practical applications, an estimator that is as free of bias as possible should be used at least for estimating the uncertainty with the sample size n_0 in the first step of the bootstrap approach. However, the analytical approach does not show such shortcomings.

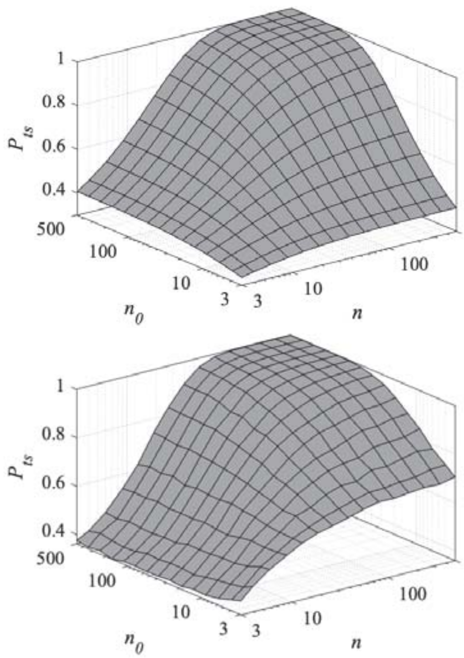

Fig. 5. P_{ts} of EoL test when considering uncertainty of prior knowledge; analytical (top), bootstrap (bottom).

4.3.2 Influence of Uncertainty on EoL Tests

Generally, the P_{ts} decreases when the uncertainty is considered in EoL tests, because the lifetime quantiles have a higher variance. Compared to the calculation without considering the uncertainty, a strong reduction takes place. The values of the P_{ts} considering the prior knowledge are limited by a maximum value, based on the sample size (or generally the uncertainty) of prior knowledge. For practical application this means that the uncertainty should be considered. However, this calculation should always be made together with a calculation without taking the uncertainty into account, because the result of this calculation then represents the maximum achievable P_{ts}.

4.3.3 Influence of Uncertainty on SR Tests

In contrast to the EoL test, there is no general tendency for the SR test to decrease or increase the P_{ts} due to the consideration of uncertainty in prior knowledge. In Figure 6 it can be seen that for very large values of P_{ts} there is a decrease, whereas for small values of P_{ts} there is an increase. This can be explained by the increased variance of the reliability, which is described by the beta distribution of the prior knowledge. With already very large reliabilities, smaller reliabilities will occur more often than larger ones due to the larger variance, relative to the reference value that would be used to calculate the P_{ts} without uncertainty. This is because if the values of the reliability already are close to the maximum value of 1, the probability that there are even larger values is small.

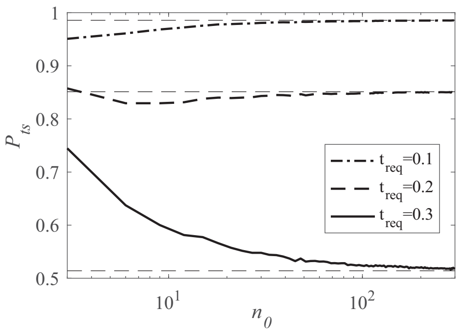

Fig. 6. P_{ts} of SR Test when considering uncertainty of prior knowledge with & $T_{H_1} = 1$ & $b_{H_1} = 3.5$

Exactly the opposing behavior is evident for small values of the P_{ts} – here, the values increase when the uncertainty is considered. This is also due to the fact that the probability of obtaining even smaller values of reliability decreases the closer the values get to the minimum value of 0. The main difference between the behavior of the P_{ts} when considering the uncertainty in the prior knowledge for the EoL test and the SR test is thus due to the restricted domain of the reliability itself. It is restricted to the interval $R \in [0, 1]$, whereas the quantity relevant in the calculation of P_{ts} for EoL tests, the test statistic τ has no such restriction. Solely the restriction of strictly positive lifetimes prevails here. However, this does not limit the behavior of the P_{ts} significantly.

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