Proceedings of the 33rd European Safety and Reliability Conference (ESREL 2023)
Edited by Mário P. Brito, Terje Aven, Piero Baraldi, Marko Čepin and Enrico Zio

doi: 10.3850/978-981-18-8071-1_P079-cd

Remaining useful life estimation of gamma degrading units characterized by a bathtub-shaped degradation rate in the presence of random effect and measurement error

Antonio Piscopo

Scuola Superiore Meridionale, Napoli, Italia. E-mail: antonio.piscopo@unina.it

Nicola Esposito

Université d'Angers/Laris, Angers, France, and Università di Napoli Federico II, Napoli, Italia. E-mail: nicola.esposito@univ-angers.fr

Bruno Castanier

Université d'Angers/Laris, Angers, France. E-mail: bruno.castanier@univ-angers.fr

Massimiliano Giorgio

Università di Napoli Federico II, Napoli, Italia. E-mail: massimiliano.giorgio@unina.it

This paper proposes a new non-homogeneous gamma process with bathtub-shaped degradation rate function that can be used in the presence of random effect and measurement error. The proposed model is not mathematically tractable. Its main features are illustrated. The probability distribution function of the remaining useful life is formulated by using a failure threshold model. The maximum likelihood estimation of the parameters of the model from perturbed data is addressed. A procedure that combines expectation-maximization algorithm and particle filter method is suggested that allows to significantly mitigate the numerical problems posed by the direct maximization of the likelihood function. The same particle filter algorithm is also adopted to compute the probability distribution function of the remaining useful life, which constitutes the core prognostic tool in condition-based maintenance. As a motivating example, the proposed model is applied to a set of real degradation data of metal-oxide-semiconductor field-effect transistors. Obtained results demonstrate the utility of the proposed model and the affordability of the suggested estimation procedure.

Keywords: bathtub-shaped degradation rate, random effect, maximum likelihood estimation, expectation-maximization algorithm, particle filter.

1. Introduction

While the degradation rate (here intended as the derivative of the mean function) of degrading technological units is typically bathtub shaped, almost all degradation models suggested in the literature are characterized by a monotonic increasing degradation rate function. In fact, real world degradation phenomena usually exhibit a first phase where the degradation rate decreases, a second phase where the degradation rate is roughly constant, and a third phase where it increases (see, e.g. Gertsbakh and Kordonskiy (1969)). In general, this circumstance does not preclude the successful application of these models because the majority of degradation data considered in the literature do not give evidence of the existence of all the mentioned phases. However, this is not the case of the metal-oxide-semiconductor field-effect transistors (MOSFETs) data represented in Figure 1, which empirical mean is clearly inverse S-shaped, so that the corresponding degradation rate function is bathtub-shaped.

Motivated by these data, Giorgio et al. (2023) suggested a new Wiener process with random effects that allows to describe degradation phenomena where the degradation rate is bathtub shaped. The Wiener process was there considered because the MOSFET dataset contains 12 negative and 32 null increments, which do not allow the direct use of monotonic increasing degradation models.

In this paper, we assume that the MOSFET degradation process is intrinsically monotonic increasing and hence that the presence of null and negative increments are due to measurement error. Accordingly, considered that neglecting the presence of the measurement error would lead to erroneous estimates of reliability and remaining useful life, a new perturbed gamma process with random effect is suggested for these data.

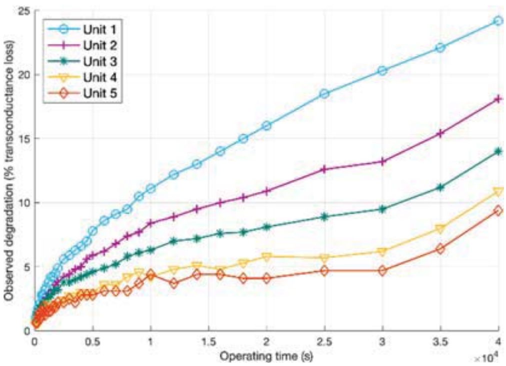

Fig. 1. The degradation paths of the five MOSFETs

Model parameters are estimated via the maximum likelihood (ML) method. Given that the likelihood function is indexed by many parameters (i.e., 8) and is not available in closed form, in order to mitigate the severe convergence issues posed by its direct maximization, a new procedure that combines the expectation-maximization (EM) method and the particle filter algorithm is suggested.

The rest of the paper is structured as follows. Section 2 presents the proposed degradation process. Sections 3 and 4 deal with the formulation of the likelihood function and remaining useful life. Sections 5 and 6 illustrate the adopted EM and particle filter algorithms. Section 6 describes the results of the application of the proposed model to the MOSFET data. Section 7 provides some concluding remarks.

2. The perturbed gamma process with random effects and bathtub-shaped degradation rate

The perturbed degradation process $\{Z(t); t \geq 0\}$ is defined as:

$$Z(t) = W(t) + \varepsilon(t) \quad (1)$$

where $\{W(t); t \geq 0\}$ is the true hidden degradation process and $\varepsilon(t)$ is a perturbing term, here intended as a measurement error. The hidden process $\{W(t); t \geq 0\}$ is assumed to be a non-homogeneous gamma process with age function:

$$\eta(t) = (a_1 t)^{b_1} + (a_2 t)^{b_2} \quad (2)$$

and scale parameter λ. The parameter a_1 is assumed to vary randomly from unit to unit. Hereinafter, we will indicate this random parameter by A_1 and its realization by a_1.

The functional form in Eq. (2), already adopted in Giorgio et al. (2023) to analyze the same MOSFET data under a Wiener process, has been specifically selected to describe degradation phenomena where the degradation rate function is bathtub shaped. In fact, under the considered hidden gamma process with random effect, given $A_1 = a_1$, the conditional probability density function (pdf) of $W(t)$ is:

$$f_{W(t)|A_1}(w|a_1) = \frac{\lambda(\lambda w)^{(a_1 t)^{b_1} + (a_2 t)^{b_2} - 1}}{\Gamma((a_1 t)^{b_1} + (a_2 t)^{b_2})} e^{-\lambda w}, \quad (3)$$

where $\Gamma(\cdot)$ is the complete gamma function (i.e., $\Gamma(u) = \int_0^\infty y^{u-1} e^{-y} \mathrm{d}y,\ u > 0$). Hence, the (conditional) mean function of $\{W(t); t \geq 0\}$ is:

$$E\{W(t)|A_1 = a_1\} = \frac{(a_1 t)^{b_1} + (a_2 t)^{b_2}}{\lambda} \quad (4)$$

and the related degradation rate function is:

$$\frac{\mathrm{d}}{\mathrm{d}t} E\{W(t)|A_1 = a_1\} = \frac{b_1 a_1 (a_1 t)^{b_1 - 1} + b_2 a_2 (a_2 t)^{b_2 - 1}}{\lambda} \quad (5)$$

which for $(b_1 - 1)/(b_2 - 1) < 0$ are inverse S-shaped and bathtub shaped, respectively.

From Eq. (3), the conditional cumulative distribution function (cdf) of $W(t)$ is:

$$F_{W(t)|A_1}(w|a_1) = \frac{\gamma((a_1 t)^{b_1} + (a_2 t)^{b_2}, \lambda w)}{\Gamma((a_1 t)^{b_1} + (a_2 t)^{b_2})}, \quad (6)$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function (i.e., $\gamma(u, x) = \int_0^x y^{u-1} e^{-y} \mathrm{d}y,\ u, x > 0$). Moreover, the conditional variance of $W(t)$ is:

$$V\{W(t)|A_1 = a_1\} = \frac{(a_1 t)^{b_1} + (a_2 t)^{b_2}}{\lambda^2}, \quad (7)$$

and the conditional pdf of $\Delta W(t, t + \tau) = W(t + \tau) - W(t)$, where $\tau > 0$, is:

$$f_{\Delta W(t,t+\tau)|A_1}(\delta|a_1) = \frac{\lambda(\lambda\delta)^{\Delta\eta(t,t+\tau) - 1}}{\Gamma(\Delta\eta(t, t + \tau))} e^{-\lambda\delta},$$

where:

$$\eta(t,t+\tau) = \eta(t+\tau) - \eta(t) = \left(a_1(t+\tau)\right)^{b_1} + \left(a_2(t+\tau)\right)^{b_2} - (a_1 t)^{b_1} - (a_2 t)^{b_2}.$$

The unit to unit variability is described by assuming that A_1 is gamma distributed with pdf:

$$g_{A_1}(a_1) = \frac{a_1^{d-1}}{c^d \Gamma(d)} e^{-a_1/c}, \quad a_1, c, d > 0.$$

Furthermore, as in Giorgio et al. (2019), it is assumed that $\varepsilon(t)$ stochastically depends on the measured degradation level $W(t)$ and that, given $W(t) = w$, has conditional pdf:

$$f_{\varepsilon(t)|W(t),A_1}(\varepsilon|w,a_1) = f_{\varepsilon(t)|W(t)}(\varepsilon|w) = \frac{\left(\alpha(w)\right)^{\beta(w)} e^{-\frac{\alpha(w)}{\varepsilon+w}}}{(\varepsilon+w)^{\beta(w)+1}\Gamma\left(\beta(w)\right)}, \quad \varepsilon \geq -w \quad (8)$$

where: $\beta(w) = \varphi w^{2-\nu} + 2$, $-\infty < \nu < +\infty$, $\varphi > 0$, and $\alpha(w) = w[\beta(w) - 1]$.

Eq. (8) gives evidence that, given $W(t) = w$, $\varepsilon(t)$ (and hence $Z(t)$) does not depend on A_1.

Finally, it is assumed that, for any $n > 1$, any $i, j = 1, \dots, n$, and any set of times $t_1, \dots, t_n$, $\varepsilon(t_j)$ and $Z(t_j)$ given $W(t_j)$ are conditionally independent of $\varepsilon(t_i), W(t_i)$, and $Z(t_i)$ $\forall i \neq j$ $(i, j = 1, \dots, n)$.

Under these assumptions, from Eqs. (1) and (8), the perturbed measurement $Z(t)$, given $W(t) = w$, is inverse gamma distributed, with conditional pdf:

$$f_{Z(t)|W(t)}(z|w) = \frac{[\alpha(w)]^{\beta(w)} z^{-\beta(w)-1}}{\Gamma\left(\beta(w)\right)} e^{-\frac{\alpha(w)}{z}},$$

conditional mean:

$$E\{Z(t)|W(t) = w\} = w,$$

and conditional variance:

$$V\{Z(t)|W(t) = w\} = \frac{w^\nu}{\varphi}.$$

The marginal pdf and cdf of $W(t)$ and $Z(t)$. are not available in closed form.

Despite this circumstance, by using the law of total mean, the following closed form expression can be obtained for the marginal mean $W(t)$:

$$E\{W(t)\} = E\left\{E\{W(t)|A_1\}\right\} = \int_0^\infty \frac{(a_1 t)^{b_1} + (a_2 t)^{b_2}}{\lambda} \frac{a_1^{d-1}}{c^d \Gamma(d)} e^{-a_1/c} \mathrm{d}a_1 = \frac{1}{\lambda}\left[(ct)^{b_1}\frac{\Gamma(d+b_1)}{\Gamma(d)} + (a_2 t)^{b_2}\right] \quad (9)$$

that, being $E\{Z(t)\} = E\left\{E\{Z(t)|W(t)\}\right\} = E\{W(t)\}$, coincides with the marginal mean of $Z(t)$. Similarly, by the law of total variance, the marginal variance of $W(t)$ is given by:

$$V\{W(t)\} = V\left\{E\{W(t)|A_1\}\right\} + E\left\{V\{W(t)|A_1\}\right\} = \frac{(ct)^{2b_1}}{\lambda^2}\frac{\Gamma(d+2b_1)}{\Gamma(d)} + \frac{(ct)^{b_1}}{\lambda^2}\frac{\Gamma(d+b_1)}{\Gamma(d)} - \frac{(ct)^{2b_1}}{\lambda^2}\left[\frac{\Gamma(d+b_1)}{\Gamma(d)}\right]^2 + \frac{(a_2 t)^{b_2}}{\lambda^2}, \quad (10)$$

and the marginal variance of $Z(t)$ is:

$$V\{Z(t)\} = V\left\{E\{Z(t)|W(t)\}\right\} + E\left\{V\{Z(t)|W(t)\}\right\} = V\{W(t)\} + E\left\{\left(W(t)\right)^\nu\right\}/\varphi \quad (11)$$

where $V\{W(t)\}$ is given in Eq. (10) and the fractional moment:

$$E\left\{\left(W(t)\right)^\nu\right\} = E\left\{E\left\{\left(W(t)\right)^\nu \middle| A_1\right\}\right\} = \int_0^\infty \frac{\Gamma((a_1 t)^{b_1} + (a_2 t)^{b_2} + \nu)}{\Gamma((a_1 t)^{b_1} + (a_2 t)^{b_2})\lambda^\nu} \frac{a_1^{d-1}}{c^d \Gamma(d)} e^{-\frac{a_1}{c}} \mathrm{d}a_1$$

in general, should be computed numerically.

It is worth to note that, under this setting, $\{Z(t); t \geq 0\}$ is not Markovian.

From Eq. (9), the degradation rate function of the (marginal) hidden process is:

$$\frac{\mathrm{d}}{\mathrm{d}t} E\{W(t)\} = \frac{b_1 t^{b_1-1} c^{b_1}}{\lambda} \frac{\Gamma(d+b_1)}{\Gamma(d)} + \frac{b_2 t^{b_2-1} a_2^{b_2}}{\lambda}.$$

which is still bathtub shaped when $(b_1 - 1)/(b_2 - 1) < 0$.

3. The likelihood function

Let us suppose that the degradation level of m units is measured by means of ad hoc inspections and that the measurements are contaminated by errors. Moreover, let us denote by $\boldsymbol{\xi} = (b_1, a_2, b_2, c, d, \nu, \varphi)$ the vector of model parameters, by $t_{i,j}$ the age of the ith unit $(i = 1, \dots, m)$ at the jth inspection epoch $(j = 1, \dots, n_i; n_i \geq 1)$, by $Z_{i,j}$ the perturbed measurement of its degradation level at $t_{i,j}$, and by $z_{i,j}$ the realization of $Z_{i,j}$. Then, the likelihood function $L(\boldsymbol{\xi}; \boldsymbol{z})$ can be formulated as:

$$L(\boldsymbol{\xi}; \boldsymbol{z}) = \prod_{i=1}^{m}\prod_{j=1}^{n_i} f_{Z_{i,j}|\mathbf{z}_{i,j-1}}\left(z_{i,j}|\mathbf{z}_{i,j-1}\right) \quad (12)$$

where $\boldsymbol{Z}_{i,j} = \{Z_{i,1}, \dots, Z_{i,j}\}$ is the set of all measurements made on the unit i until $t_{i,j}$, $\boldsymbol{z}_{i,j} = \{z_{i,1}, \dots, z_{i,j}\}$ is its realization, $\boldsymbol{Z} = \{\boldsymbol{Z}_{1,n_1}, \dots, \boldsymbol{Z}_{i,n_m}\}$ is the entire set of perturbed measurements, $\boldsymbol{z} = \{\boldsymbol{z}_{1,n_1}, \dots, \boldsymbol{z}_{i,n_m}\}$ is its realization, $t_{i,0} = 0$, $\boldsymbol{Z}_{i,0}$ and $\boldsymbol{z}_{i,0}$ are the empty set, and $f_{Z_{i,1}|\boldsymbol{Z}_{i,0}}(z_{i,1}|\boldsymbol{z}_{i,0}) = f_{Z_{i,1}}(z_{i,1})$.

Unfortunately, under the proposed model, the likelihood function in Eq. (12) is not available in closed form. However, it can be efficiently computed by using the particle filter algorithm described in Section 6. The ML estimate $\hat{\boldsymbol{\xi}}$ of $\boldsymbol{\xi}$ is the value of $\boldsymbol{\xi}$ that maximizes the likelihood function over the parameter space. In fact, even by using the particle filter, the direct maximization of the likelihood function poses serious convergence issues. For this reason, in this paper the ML estimate of $\boldsymbol{\xi}$ is computed by using the EM algorithm described in Section 5.

4. The cdf of the RUL

We assume that a unit fails when its degradation level passes for the first time a fixed threshold, say w_M. Thus, the useful life X of a unit is defined as:

$$X = \inf\{x: W(x) > w_M\}$$

and its remaining useful life $RUL(t)$ at the operating time t is defined as:

$$RUL(t) = \max\{0, X - t\}.$$

Moreover, we assume that failures are not self-announcing and (thus) that perturbed measurements alone do not allow to determine with certainty whether a unit is failed.

Under these assumptions, given that the degradation process described in Section 2 is monotonic increasing, the conditional cdf of $RUL(t)$ given $\boldsymbol{Z}(t) = \boldsymbol{z}(t)$ can be expressed as:

$$\begin{aligned} &F_{RUL(t)|\boldsymbol{Z}(t)}(\tau|\boldsymbol{z}(t)) \\ &= P[RUL(t) \le \tau|\boldsymbol{Z}(t) = \boldsymbol{z}(t)] \\ &= P[W(t+\tau) > w_M|\boldsymbol{Z}(t) = \boldsymbol{z}(t)] \\ &= 1 - F_{W(t+\tau)|\boldsymbol{Z}(t)}(w_M|\boldsymbol{z}(t)) \\ &= 1 - \int_0^{w_M} F_{\Delta W(t,t+\tau)}(w_M - w) \times \\ &\qquad \times f_{W(t)|\boldsymbol{Z}(t)}(w|\boldsymbol{z}(t)) \cdot dw \qquad (13), \end{aligned}$$

where $\boldsymbol{Z}(t) = \{Z(t_j); j \ge 1, t_j \le t\}$ is the set of measurements gathered until time t and $\boldsymbol{z}(t) = \{z(t_j); j \ge 1, t_j \le t\}$ is its realization.

It is worth emphasizing that, given that failures are not self-announcing it could result $F_{RUL(t)|\boldsymbol{Z}(t)}(0|\boldsymbol{z}(t)) > 0$. Moreover, for $t < t_1$, given that $\boldsymbol{Z}(t)$ is the empty set, $F_{W(t+\tau)|\boldsymbol{Z}(t)}(w_M|\boldsymbol{z}(t))$ reduces to $F_{W(t+\tau)}(w_M)$.

From Eq. (13) the conditional mean $MRUL(t|\boldsymbol{Z}(t) = \boldsymbol{z}(t))$ of $RUL(t)$ can be formulated as:

$$\begin{aligned} &MRUL(t|\boldsymbol{Z}(t) = \boldsymbol{z}(t)) \\ &\quad = \int_0^{\infty} \left(1 - F_{RUL(t)}(\tau|\boldsymbol{z}(t))\right) \cdot \mathrm{d}\tau \\ &\quad = \int_0^{\infty} F_{W(t+\tau)|\boldsymbol{Z}(t)}(w_M|\boldsymbol{z}(t)) \cdot \mathrm{d}\tau. \qquad (14) \end{aligned}$$

Under the proposed model, $F_{W(t+\tau)|\boldsymbol{Z}(t)}(w_M|\boldsymbol{z}(t))$, $F_{RUL(t)|\boldsymbol{Z}(t)}(\tau|\boldsymbol{z}(t))$, and $MRUL(t|\boldsymbol{Z}(t) = \boldsymbol{z}(t))$ cannot be obtained in closed form. However, they can be numerically computed via the particle filter presented in Section 6.

5. EM algorithm

The EM (Dempster et al. (1974)) is an iterative algorithm used for retrieving ML estimates in the presence of missing values and/or incomplete observations. The EM consists of two steps, an E-step (expectation step) and an M-step (maximization step), which are iterated until a convergence condition is attained.

In this paper, we consider as missing data:

- the value $\boldsymbol{a}_1 = \{a_{1,1}, \dots, a_{1,m}\}$ of the random scale parameter $\boldsymbol{A}_1 = \{A_{1,1}, \dots, A_{1,m}\}$, and
- the value $\boldsymbol{w} = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$ of the set $\boldsymbol{W} = \{\boldsymbol{W}_1, \dots, \boldsymbol{W}_m\}$ of the true degradation levels of the m units at the measurement times, where $\boldsymbol{W}_i = \{W_{i,1}, \dots, W_{i,n_i}\}$, $\boldsymbol{w}_i = \{w_{i,1}, \dots, w_{i,n_i}\}$, $W_{i,j} = W(t_{i,j})$, and $w_{i,j}$ is its realization.

Whereas the observed data consists in the realization $\boldsymbol{z} = \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_m\}$ of the set of perturbed measurements $\boldsymbol{Z} = \{\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_m\}$, where $\boldsymbol{Z}_i = \{Z_{i,1}, \dots, Z_{i,n_i}\}$, and $\boldsymbol{z}_i = \{z_{i,1}, \dots, z_{i,n_i}\}$.

With this setting, under the proposed model, the complete likelihood function of missing and observed data is:

$$\begin{aligned} L(\boldsymbol{\xi}; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{a}_1) = \prod_{i=1}^{m} & f_{\boldsymbol{Z}_i|\boldsymbol{W}_i}(\boldsymbol{z}_i; \boldsymbol{w}_i) \\ & \times f_{\boldsymbol{W}_i|A_{1,i}}(\boldsymbol{w}_i|a_{1,i}) \cdot g_{A_{1,i}}(a_{1,i}) \end{aligned}$$

where:

$$f_{\boldsymbol{Z}_i|\boldsymbol{W}_i}(\boldsymbol{z}_i|\boldsymbol{w}_i) = \prod_{i=1}^{n_i} \frac{[\alpha(w_{i,j})]^{\beta(w_{i,j})} z_{i,j}^{-\beta(w_{i,j})-1}}{\Gamma\left(\beta(w_{i,j})\right)} e^{-\frac{\alpha(w_{i,j})}{z_{i,j}}},$$

$$f_{\boldsymbol{W}_i|A_{1,i}}(\boldsymbol{w}_i|a_{1,i}) = \prod_{j=1}^{n_i} \frac{\lambda^{\Delta\eta_{i,j}(a_{1,i})} \delta_{i,j}^{\Delta\eta(t_{i,j-1},t_{i,j})-1}}{\Gamma\left(\Delta\eta_{i,j}(a_{1,i})\right)} e^{-\lambda\delta_{i,j}},$$

and:

$$g_{A_1}(a_{1,i}) = \frac{a_{1,i}^{d-1}}{c^d \Gamma(d)} e^{-a_{1,i}/c},$$

with $\Delta\eta_{i,j}(a_{1,i}) = \Delta\eta(t_{i,j-1}, t_{i,j}) = (a_{1,i}t_{i,j})^{b_1} + (a_2 t_{i,j})^{b_2} - (a_{1,i}t_{i,j-1})^{b_1} - (a_2 t_{i,j-1})^{b_2}$ and $\delta_{i,j} = w_{i,j} - w_{i,j-1}, w_{i,0} = 0\ \forall i$.

Accordingly, the complete log-likelihood function (i.e., $l(\cdot\,;\cdot) = \ln L(\cdot\,;\cdot)$) results in:

$$l(\boldsymbol{\xi}; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{a}_1) = l_E(\nu, \varphi; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{a}_1) + l_R(c, d; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{a}_1) + l_H(\lambda, a_2, b_1, b_2; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{a}_1),$$

where

$$l_E(\nu, \varphi; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{a}_1) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \beta(w_{i,j}) \ln\left(\alpha(w_{i,j})\right) - \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{\alpha(w_{i,j})}{z_{i,j}} - \sum_{i=1}^{m}\sum_{j=1}^{n_i} \ln\left(\Gamma\left(\beta(w_{i,j})\right)\right) - \sum_{i=1}^{m}\sum_{j=1}^{n_i} \left(\beta(w_{i,j}) + 1\right) \ln(z_{i,j})$$

is indexed only by the parameters ν and φ,

$$l_R(c, d; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{a}_1) = (d-1)\sum_{i=1}^{m} \ln(a_{1,i}) - \frac{1}{c}\sum_{i=1}^{m} a_{1,i} - m\ln\left(\Gamma(d)\right) - md\ln(c)$$

is indexed only by the parameters c and d, and

$$l_H(\lambda, a_2, b_1, b_2; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{a}_1) = \ln(\lambda)\sum_{i=1}^{m} \eta_{i,n_i}(a_{1,i}) - \lambda\sum_{i=1}^{m} w_{i,n_i} + \sum_{i=1}^{m}\sum_{j=1}^{n_i} \left(\Delta\eta_{i,j}(a_{1,i}) - 1\right) \ln(\delta_{i,j}) - \sum_{i=1}^{m}\sum_{j=1}^{n_i} \ln\left(\Gamma\left(\Delta\eta_{i,j}(a_{1,i})\right)\right),$$

is indexed only by the parameters λ, a_2, b_1, and b_2, with $\eta_{i,n_i}(a_{1,i}) = \sum_{j=1}^{n_i} \Delta\eta_{i,j}(a_{1,i}) = \eta(t_{i,n_i}) = (a_{1,i}t_{i,n_i})^{b_1} + (a_2 t_{i,n_i})^{b_2}$.

After h iterations, the E-step consists in computing the conditional mean:

$$Q(\boldsymbol{\xi}|\boldsymbol{\xi}^{(h)}) = E\{l(\boldsymbol{\xi}; \boldsymbol{z}, \boldsymbol{W}, \boldsymbol{A}_1)|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}\} = Q_E(\nu, \varphi|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}) + Q_R(c, d|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}) + Q_H(\lambda, a_2, b_1, b_2|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}),$$

where $\boldsymbol{\xi}^{(h)}$ indicates the estimate of $\boldsymbol{\xi}$ obtained by performing the hth M-step and the functions $Q_E(\cdot)$, $Q_R(\cdot)$, and $Q_H(\cdot)$ can be expressed as:

$$Q_E(\nu, \varphi|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} E\left\{\beta(W_{i,j}) \ln\left(\alpha(W_{i,j})\right) \middle| \boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}\right\} - \sum_{i=1}^{m}\sum_{j=1}^{n_i} E\{\beta(W_{i,j}) + 1|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}\} \ln(z_{i,j}) - \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{E\{\alpha(W_{i,j})|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}\}}{z_{i,j}} - \sum_{i=1}^{m}\sum_{j=1}^{n_i} E\left\{\log\left(\Gamma\left(\beta(W_{i,j})\right)\right) \middle| \boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}\right\},$$

$$Q_R(c, d|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}) = -md\ln(c) - m\ln\left(\Gamma(d)\right) - \frac{1}{c}\sum_{i=1}^{m} E\{A_{1,i}|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}\} + (d-1)\sum_{i=1}^{m} E\{\ln(A_{1,i})|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}\},$$

and

$$Q_H(\lambda, a_2, b_1, b_2|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}) = \ln(\lambda)\sum_{i=1}^{m} E\{\eta_{i,n_i}(A_{1,i})|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}\} - \lambda\sum_{i=1}^{m} E\{W_{i,n_i}|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}\} + \sum_{i=1}^{m}\sum_{j=1}^{n_i} E\{\Delta\eta_{i,j}(A_{1,i}) \ln(\Delta W_{i,j})\,|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}\} - \sum_{i=1}^{m}\sum_{j=1}^{n_i} E\{\ln(\Delta W_{i,j})\,|\boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}\}$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n_i} E\left\{\ln\left(\Gamma\left(\Delta\eta_{i,j}(A_{1,i})\right)\right)\middle|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\right\},$$

where $\Delta W_{i,j} = W_{i,j} - W_{i,j-1}$ and $W_{i,0} = 0\ \forall i$.

In these equations, the presence of $\boldsymbol{\xi}^{(h)}$ on the right side of the conditional bar indicates that the parameters of the conditional distribution of $\boldsymbol{W}$ and $\boldsymbol{A}_1$ given $\boldsymbol{Z}=\boldsymbol{z}$ used to perform the expectations, are set to $\boldsymbol{\xi}^{(h)}$. The conditional expectations are computed by using the particle filter approach described in Section 6.

The M-step consists in maximizing $Q(\boldsymbol{\xi}|\boldsymbol{\xi}^{(h)})$ with respect to $\boldsymbol{\xi}$. The value of $\boldsymbol{\xi}$ that maximizes $Q(\boldsymbol{\xi}|\boldsymbol{\xi}^{(h)})$ (i.e., the new estimate of $\boldsymbol{\xi}$) is denoted by $\boldsymbol{\xi}^{(h+1)}$.

In this paper, the iterative procedure is stopped when the absolute relative difference:

$$\left|\frac{\ln\left(L(\boldsymbol{\xi}^{(h+1)};\boldsymbol{z})\right) - \ln\left(L(\boldsymbol{\xi}^{(h)};\boldsymbol{z})\right)}{\ln\left(L(\boldsymbol{\xi}^{(h)};\boldsymbol{z})\right)}\right|,$$

where $L(\cdot\,;\cdot)$ is the likelihood function in Eq. (12), drops below an assigned value.

Note that, the values of λ and c that maximize $Q_H(\cdot)$ and $Q_E(\cdot)$ can be computed as:

$$\lambda^{(h+1)} = \frac{\sum_{i=1}^{m} E\{\eta_{i,n_i}(A_{1,i})|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\}}{\sum_{i=1}^{m} E\{W_{i,n_i}|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\}}$$

and

$$c^{(h+1)} = \frac{\sum_{i=1}^{m} E\{A_{1,i}|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\}}{md^{(h+1)}},$$

respectively.

The estimates of the other parameters should be retrieved numerically. In particular, $d^{(h+1)}$ can be obtained by maximizing with respect to d the function $Q_R(c^*(d), d|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)})$ where $c^*(d) = \sum_{i=1}^{m} E\{A_{1,i}|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\}/(md)$.

6. The particle filter algorithm

The particle filter algorithm (Doucet and Johansen (2011)) allows, for an assigned value of $\boldsymbol{\xi}$ (e.g., the value $\boldsymbol{\xi}^{(h)}$ obtained after the hth M-step of the EM), to generate N pseudorandom realizations of $A_{1,i}$ and $\boldsymbol{W}_i$ given $\boldsymbol{Z}_i = \boldsymbol{z}_i$.

The method consists of the following two steps, which must be iterated n_i times.

- Step 1 (prediction step), jth iteration: for any $k = 1, \dots, N$, set $A_{1,i}$ to ${}^{j-1}_{k}a_{1,i}$, generate a pseudorandom realization ${}_{k}\Delta w_{i,j}$ of $\Delta W_{i,j}$, compute ${}^{j-1}_{k}w_{i,j} = {}_{k}\Delta w_{i,j} + {}^{j-1}_{k}w_{i,j-1}$, and append it to the particle vector ${}^{j-1}_{k}a_{1,i}, {}^{j-1}_{k}w_{i,1}, \dots, {}^{j-1}_{k}w_{i,j-1}$ defined at the $(j-1)$th iteration. The output of this prediction step is a set of N vectors:

$$\begin{matrix} {}^{j-1}_{1}a_{1,i}, {}^{j-1}_{1}w_{i,1}, \dots, {}^{j-1}_{1}w_{i,j-1}, {}^{j-1}_{1}w_{i,j} \\ \vdots \\ {}^{j-1}_{N}a_{1,i}, {}^{j-1}_{N}w_{i,1}, \dots, {}^{j-1}_{N}w_{i,j-1}, {}^{j-1}_{N}w_{i,j}\ , \end{matrix}$$

which will be referred to as particles.

- Step 2 (update step), jth iteration: for any $k = 1, \dots, N$, compute the importance weight of the kth particle as:

$${}_{k}q_{i,j} = \frac{f_{Z_{i,j}|W_{i,j}}\left(z_{i,j}\middle|{}^{j-1}_{k}w_{i,j}\right)}{\sum_{k=1}^{N} f_{Z_{i,j}|W_{i,j}}\left(z_{i,k}\middle|{}^{j-1}_{k}w_{i,j}\right)},$$

resample the set of particles:

$$\begin{matrix} {}^{j-1}_{1}a_{1,i}, {}^{j-1}_{1}w_{i,1}, \dots, {}^{j-1}_{1}w_{i,j-1}, {}^{j-1}_{1}w_{i,j} \\ \vdots \\ {}^{j-1}_{N}a_{1,i}, {}^{j-1}_{N}w_{i,1}, \dots, {}^{j-1}_{N}w_{i,j-1}, {}^{j-1}_{N}w_{i,j} \end{matrix}$$

according to their importance weights and rename the new particles as:

$$\begin{matrix} {}^{j}_{1}a_{1,i}, {}^{j}_{1}w_{i,1}, \dots, {}^{j}_{1}w_{i,j-1}, {}^{j}_{1}w_{i,j} \\ \vdots \\ {}^{j}_{N}a_{1,i}, {}^{j}_{N}w_{i,1}, \dots, {}^{j}_{N}w_{i,j-1}, {}^{j}_{N}w_{i,j}\ . \end{matrix}$$

In the first prediction step (i.e., for $j = 1$), initialize the algorithm by drawing a pseudorandom sample of size N from the joint distribution of $A_{1,i}$ and $W_{i,1}$, denote its elements by $({}_{1}a_{1,i}, {}_{1}w_{i,1}), \dots, ({}_{1}a_{1,i}, {}_{N}w_{i,1})$, and define the particles as:

$$\begin{matrix} {}^{0}_{1}a_{1,i}, {}^{0}_{1}w_{i,1} = {}_{1}a_{1,i}, {}_{1}w_{i,1} \\ \vdots \\ {}^{0}_{N}a_{1,i}, {}^{0}_{N}w_{i,1} = {}_{N}a_{1,i}, {}_{N}w_{i,1}. \end{matrix}$$

The particle ${}^{j-1}_{k}a_{1,i}, {}^{j-1}_{k}w_{i,1}, \dots, {}^{j-1}_{k}w_{i,j-1}, {}^{j-1}_{k}w_{i,j}$ should be intended as a realization of $A_{1,i}$ and $\boldsymbol{W}_{i,j}$ given $\boldsymbol{Z}_{i,j-1} = \boldsymbol{z}_{i,j-1}$, and the particle ${}^{j}_{k}a_{1,i}, {}^{j}_{k}w_{i,1}, \dots, {}^{j}_{k}w_{i,j-1}, {}^{j}_{k}w_{i,j}$ as a realization of $A_{1,i}$ and $\boldsymbol{W}_{i,j}$ given $\boldsymbol{Z}_{i,j} = \boldsymbol{z}_{i,j}$. The conditional pdfs that are needed to compute the likelihood function in Eq. (12) can be approximated as:

$$f_{Z_{i,j}|\boldsymbol{Z}_{i,j-1}}(z_{i,j}|\boldsymbol{z}_{i,j-1})$$

$$\cong \frac{\sum_{k=1}^{N} f_{Z_{i,j}|W_{i,j}}\left(z_{i,j}\middle|{}_{k}^{j-1}w_{i,j}\right)}{N}$$

where ${}_{k}^{j-1}w_{i,j}$ is the last component of the particle ${}_{k}^{j-1}a_{1,i}, {}_{k}^{j-1}w_{i,1}, \dots, {}_{k}^{j-1}w_{i,j-1}, {}_{k}^{j-1}w_{i,j}$ generated at jth prediction step.

Similarly, for example, the conditional mean of a function $g(A_i, \boldsymbol{W}_i)$ of A_i and $\boldsymbol{W}_i$, given $\boldsymbol{Z}_i = \boldsymbol{z}_i$ and $\boldsymbol{\xi}$, can be computed as:

$$E\{g(A_i, \boldsymbol{W}_i)|\boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{\xi}\} \cong \frac{\sum_{k=1}^{N} g\left({}_{k}^{n_i}a_{1,i}, {}_{k}^{n_i}\boldsymbol{w}_i\right)}{N}$$

where ${}_{k}^{n_i}a_{1,i}$ and ${}_{k}^{n_i}\boldsymbol{w}_i = \{{}_{k}^{n_i}w_{i,1}, \dots, {}_{k}^{n_i}w_{i,n_i-1}, {}_{k}^{n_i}w_{i,n_i}\}$ are computed under the suggested degradation model with random effect with parameter vector set to $\boldsymbol{\xi}$. Obviously, the quality of these approximations improves with N.

This particle filter algorithm is also used to compute the ML estimate of the cdf of $RUL(t)$, $F_{RUL(t)}(\tau|\boldsymbol{z}(t))$, given in Eq. (13), and the ML estimate of the $MRUL(t)$, given in Eq. (14). In particular, by using the notations introduced in Section 4, if $t_l \leq t < t_{l+1}$, so that the set $\boldsymbol{Z}(t) = \{Z(t_j); j \geq 1, t_j \leq t\}$ contains l perturbed measurements of the degradation level of a certain unit, given a pseudorandom sample of size N from $W(t_l)|\boldsymbol{Z}(t) = \boldsymbol{z}(t)$, say ${}_{1}^{l}w_l, \dots, {}_{N}^{l}w_l$, the ML estimate of $F_{RUL(t)}(\tau|\boldsymbol{z}(t))$ can be computed as:

$$F_{RUL(t)}\left(\tau\middle|\boldsymbol{z}(t)\right) \cong 1 - \frac{\sum_{k=1}^{N} F_{W(t+\tau)|W(t_l)}\left(w_M\middle|{}_{k}^{l}w_l\right)}{N}.$$

where both the parameters of the perturbed model used to generate the particles and the parameters of the conditional cdf $F_{W(t+\tau)|W(t_l)}(\cdot|\cdot)$ are set to their ML estimates.

7. Numerical example

In this section we report and briefly discuss the results obtained by applying the proposed model to the MOSFET data depicted in Figure 1. The complete dataset is available in Lu et al. (1997) and Giorgio et al. (2023). The data consist of measurements of percent transconductance loss of $m = 5$ MOSFETs. There are $n = 35$ measurements for each MOSFET. The observation times $t_1, \dots, t_{35}$ are expressed in seconds and are the same for all the units. The last observation is performed at $t_{35} = 40{,}000$ seconds. A MOSFET is assumed to fail when its percent transconductance loss passes the threshold level $w_M = 25\%$. All the estimates reported in this section have been obtained by using the EM algorithm and/or the particle filter approach described in Sections 5 and 6.

Here we show only the results obtained by using the model where the parameter ν of the error term $\varepsilon(t)$ is set to 0, which according to the Akaike information criterion (Akaike (1974)), should be preferred to the complete one. Under this model, we have obtained the following ML estimates of the remaining parameters: $\hat{c} = 3.28$, $\hat{d} = 1.08$, $\hat{b}_1 = 0.453$, $\hat{a}_2 = 3.79 \cdot 10^{-5}$, $\hat{b}_2 = 9.30$, $\hat{\lambda} = 15.7$, and $\hat{\varphi} = 35.7$.

The ML estimates of $E\{Z(t)\} = E\{W(t)\}$ are depicted in Figure 2 together with the empirical estimates of $E\{Z(t)\}$ at the inspection points. The plot shows that the proposed model fits very well the empirical mean.

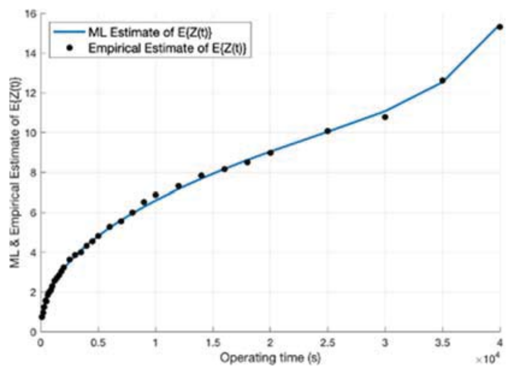

Fig. 2. ML and empirical estimates of the degradation mean $E\{Z(t)\} = E\{W(t)\}$.

In Figure 3 there are the ML estimates of $V\{Z(t)\}$ and $V\{W(t)\}$ and the empirical estimates of $V\{Z(t)\}$ at the inspection times. This figure shows that the estimated model fits sufficiently well also the empirical variance. In addition, it also shows that the ML estimates of $V\{Z(t)\}$ and $V\{W(t)\}$ are very close to each other, a circumstance which indicates that in this application the measurement error is relatively small. Note that, when $\nu = 0$, from Eq. (11), $V\{Z(t)\} = V\{W(t)\} + 1/\varphi$.

Finally, Figure 4 shows the ML estimates of the conditional cdf, $F_{RUL(t)}(\tau|\boldsymbol{z}(t))$), of the RUL of all the MOSFETs, at the last inspection epoch $t_{35} = 40{,}000$ seconds, given all the available perturbed measurements. The values of the corresponding estimated MRULs are reported in Table 1.

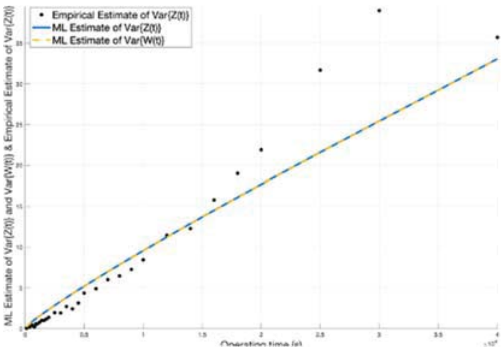

Fig. 3. ML and empirical estimates of $V\{Z(t)\}$ at the inspection times and the ML estimates of $V\{W(t)\}$ and $V\{Z(t)\}$.

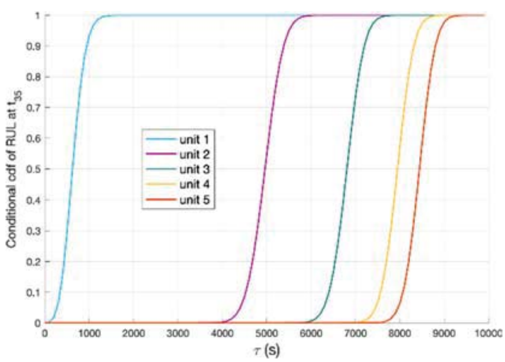

Fig. 4. ML estimates of $F_{RUL(t_{35})}(\tau|\boldsymbol{z}(t_{35}))$, at t_{35} = 40,000 seconds, for $w_{\mathrm{M}} = 25\%$.

Table 1. ML estimate of the MOSFETs $MRUL(t_{35}|\boldsymbol{Z}_i = \boldsymbol{z}_i)$, $i = 1, \dots, 5$, (in seconds), for $w_{\mathrm{M}} = 25\%$.

Unit	1	2	3	4	5
MRUL	635	4957	6803	7943	8438

8. Conclusions

Motivated by a real set of MOSFET degradation data, in this paper, we have proposed a new gamma process-based degradation model that can be used to describe phenomena characterized by bathtub-shaped degradation rate functions in the presence of random effect and measurement error. The features of the model have been illustrated. Hence, the maximum likelihood estimation of model parameters has been addressed and a procedure that combines a particle filter and an expectation-maximization algorithm has been suggested which allows to efficiently retrieve the maximum likelihood estimates of model parameters from perturbed data. The probability distribution function and the remaining useful life of the units are formulated by using a failure threshold model.

The model is finally applied to the MOSFET data. Obtained results demonstrate the utility of the proposed model and the feasibility of the suggested estimation procedure.

Acknowledgments

This research activity was supported by Università di Napoli Federico II in the frame of the international agreement between Dipartimento di Ingegneria Industriale and Polytech Angers (codice identificativo 000011–ALTRI-2021-M-GIORGIO_001_001, prot. 40162 del 20/4/2021), by the Université Franco Italienne within the frame of the chapitre 2 of Vinci project (subvention N° C2-221), and by the WISE project of the Région Pays de la Loire (France).

References

Gertsbakh I.B., K. B. Kordonskiy (1969). Models of Failure. Springer-Verlag, Berlin.

Giorgio M., A. Piscopo, and G. Pulcini (2023). A new Wiener process with bathtub-shaped degradation rate in the presence of random effects, *Applied Stochastic Models in Business and Industry*, 1-24, DOI: 10.1002/asmb.2749

Giorgio M., A. Mele, and G. Pulcini (2019). A perturbed gamma degradation process with degradation dependent non-Gaussian measurement errors, *Applied Stochastic Models in Business and Industry 35(2),* 198-210.

Dempster A. P., N. M. Laird, and D. B. Rubin (1977), Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society Series B 39(1)*, 1-22

Doucet A. and A. M. Johansen (2011). A tutorial on particle filtering and smoothing: fifteen years later. In: Crisan, D. and Rozovskii, B., Eds., *The Oxford Handbook of Nonlinear Filtering*, Oxford University Press, New York, 656-704.

Lu J.C., J. Park, and Q. Yang (1997). Statistical inference of a time-to-failure distribution derived from linear degradation data. *Technometrics 39(4)*, 391-400.

Akaike H. (1974). A new look at the statistical model identification. *IEEE Transactions on Automatic Control 19(6)*, 716–723.