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Bayesian inference for the bounded transformed gamma process

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Very recently, a novel stochastic process model, called the bounded transformed gamma process, has been proposed to describe bounded degradation phenomena, where the degradation level can not exceed a given upper bound, due to inherent features of the degradation causing mechanism. In this paper, a Bayesian estimation procedure is developed and illustrated for such a stochastic process, which uses prior information on the upper bound and on other physical characteristics of the degradation phenomenon under observation. Several experimental scenarios are considered and, for each of them, specific prior distributions are suggested which allow to convey into the inferential procedure the different information the analyst is assumed to possess. A Monte Carlo Markov Chain method is developed to estimate the process parameters and some functions thereof, such as the mean degradation level, the residual reliability of a unit, and to predict the future degradation growth. Finally, the proposed procedure is validated on a set of real data containing wear measurements in different time instants of the liners of an 8-cylinder Diesel engine for marine propulsion.

Keywords: Transformed gamma process, bounded degradation phenomena, Bayesian estimation, Monte Carlo Markov Chain method, remaining useful life, residual reliability.

1. Introduction

Most of the stochastic models adopted to describe degradation phenomena of technological units assume that their degradation level can increase infinitely. However, this assumption is often not realistic, because many real-world degradation phenomena are intrinsically bounded above (see, e.g., Giorgio et al. (2015a), Ling et al. (2015), and Deng and Pandey (2016)), if only because technological units have finite size.

Motivated by this argument, very recently Fouladirad et al. (2023) proposed a bounded version of the transformed gamma process (Giorgio et al. (2015a)), named bounded transformed gamma process (BTGP), expressly conceived to describe monotonic increasing degradation phenomena where the degradation level can not exceed an upper limit U. A distinguishing feature of the BTGP, with respect to other existing bounded models (see, e.g., Giorgio et al. (2015a), Ling et al. (2015), and Deng and Pandey (2016)) is that the BTGP treats the upper limit U as an unknown parameter, which has to be estimated from data.

Maximum likelihood estimation of the BTGP parameters has been discussed in Fouladirad et al. (2023).

In this paper, a Bayesian estimation procedure is proposed that allows to directly incorporate into the inferential process various types of technological information on the observed degradation phenomenon that are often available in practical settings. In particular, the paper focuses on prior information formulated in terms of: *a*) the upper bound U of the degradation process, and *b*) the possible presence of an inflection point in the mean degradation function. Bayesian inference on process parameters and on several useful functions of the parameters themselves, such as the mean remaining useful life and the residual reliability, is based on Monte Carlo Markov Chain (MCMC) methods. The prediction of the degradation process increment over a future time interval is also performed. The proposed approach is finally verified on a set of real data, originally given in Giorgio et al. (2015b), containing wear measurements of the liners of an 8-cylinder engine of a cargo ship, already analyzed in Fouladirad et al. (2023).

2. The bounded transformed gamma process

The bounded transformed gamma process (BTGP) $\{W(t);\ t \geq 0\}$ is an asymptotically bounded above monotonic increasing Markovian process with dependent increments (Fouladirad et al. (2023)). Being Markovian, the BTGP is completely defined by the conditional probability density function (pdf) of its increment $\Delta W(t, t + \tau)$ over the time interval $(t, t+\tau)$, given the current state $W(t) = w_t$, that is defined as:

$$f_{\Delta W(t,t+\Delta t)|W(t)}(\delta|w_t) = g'(w_t+\delta)\frac{[\Delta g(w_t, w_t+\ \delta)]^{\Delta\eta(t,t+\tau)-1}}{\Gamma[\Delta\eta(t,t+\tau)]} \times e^{-\Delta g(w_t,w_t+\delta)},\quad 0<\delta<U-w_t\ ,\quad (1)$$

where:

- U is the upper bound of $\{W(t);\ t \geq 0\}$,
- $g(w)$ is a non-negative, monotone increasing and differentiable function of the degradation level w, defined over the domain $0 \leq w < U$, with $g(0) = 0$ and $\lim_{w\to U} g(w) = \infty$,
- $g'(w_t + \delta)$ is the first derivative of $g(\cdot)$ evaluated at $w_t + \delta$,
- $\Delta g(w_t, w_t + \delta) = g(w_t + \delta) - g(w_t)$,
- $\eta(t)$ is a non-negative, monotone increasing function, defined on the domain $0 \leq t < \infty$, with $\eta(0) = 0$ and $\lim_{t\to\infty} \eta(\eta) = \infty$,
- $\Delta\eta(t, t + \tau) = \eta(t + \tau) - \eta(t)$, and
- $\Gamma[\Delta\eta(t, t + \tau)]$ is the complete gamma function.

The functions $\eta(\cdot)$ and $g(\cdot)$ are called age and bounded state function. From (1), the conditional cumulative distribution function (Cdf) of $\Delta W(t, t + \tau)$, given $W(t) = w_t$, is:

$$F_{\Delta W(t,t+\tau)|W(t)}(\delta|w_t) = \frac{\gamma\big(\Delta g(w_t, w_t+\delta); \Delta\eta(t,t+\tau)\big)}{\Gamma\big(\Delta\eta(t,t+\tau)\big)},\quad 0<\delta<U, (2)$$

where $\gamma(x; a) = \int_0^x z^{a-1}e^{-z}\,\mathrm{d}z$ is the (lower) incomplete gamma function. From (1) and (2), by replacing $\Delta g(w_t, w_t + \delta)$, $g'(w_t + \delta)$, and $\Delta\eta(t, t + \tau)$ by $g(w)$, $g'(w)$, and $\eta(t)$, respectively, the following pdf and the Cdf of the degradation level $W(t)$ of a new unit, being $W(t) = \Delta W(0, t)$, are readily obtained:

$$f_{W(t)}(w) = g'(w)\frac{[g(w)]^{\eta(t)-1}}{\Gamma\big(\eta(t)\big)}e^{-g(w)},\quad 0<w<U,\quad (3)$$

$$F_{W(t)}(w) = \frac{\gamma\big(g(w);\eta(t)\big)}{\Gamma\big(\eta(t)\big)},\quad 0<w<U\quad (4)$$

respectively. In the following, the BTGP with:

$$g(w) = \beta\,\frac{w}{U-w}\quad,\quad \eta(t) = (t/a)^b,\quad (5)$$

and $g'(w) = \beta\frac{U}{(U-w)^2}$.

will be adopted, which in Fouladirad et al. (2023) it showed to provide, within the classical approach, a good fit for the wear data that will be the object of the numerical application discussed in Section 6. From (5), we have that:

$$\Delta\eta(t, t+\tau) = \left(\frac{t+\tau}{a}\right)^b - \left(\frac{t}{a}\right)^b,$$

$$\Delta g(w_t, w_t+\delta) = \beta\,\left(\frac{w_t+\delta}{U-w_t-\delta} - \frac{w_t}{U-w_t}\right).\quad (6)$$

The mean and variance of $W(t)$, as well as the conditional mean and variance of $\Delta W(t, t + \tau)$, given $W(t) = w_t$, are not in closed form, and require univariate numerical integrations:

$$E\{W(t)\} = \int_0^U w\, f_{W(t)}(w)\mathrm{d}w,\quad (7)$$

$$V\{W(t)\} = \int_0^U w^2\, f_{W(t)}(w)\mathrm{d}w - E^2\{W(t)\}, (8)$$

$$E\{\Delta W(t, t+\tau)|W(t) = w_t\} = \int_0^{U-w_t} \delta\, f_{\Delta W(t,t+\tau)|W(t)}(\delta|w_t)\,\mathrm{d}\delta\ ,\quad (9)$$

$$V\{\Delta W(t,t+\tau)|W(t)=w_t\} = \int_0^{U-w_t} \delta^2 \, f_{\Delta W(t,t+\tau)|W(t)}(\delta|w_t)\, \mathrm{d}\delta - E^2\{\Delta W(t,t+\tau)|W(t)=w_t\}. \quad (10)$$

Figure 1 shows the behavior of the mean function $E\{W(t)\}$ of the BTGP with the bounded state and age functions in (5), for $U = 10$, $a = 1$, $\beta = 1$, and selected values of b, say $b = 0.5$, 0.9, 1.0, 1.1, 2.0, and 3.0. When $b > 1$ (i.e., when $\eta(t)$ is convex), the mean function has an inflection point. Otherwise, the mean function is concave, and its first derivative with respect to t decreases monotonically with t. For b close to 1, the mean function initially grows almost linearly.

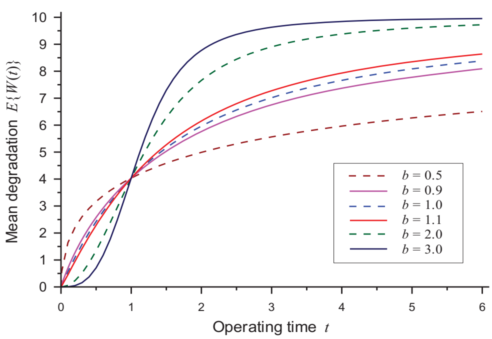

Fig 1. The behavior of the mean function of the BTGP, for $U = 10$, $a = 1$, $\beta = 1$, and selected values of b, say $b = 0.5$, 0.9, 1.0, 1.1, 2.0, and 3.0.

Degrading units are conventionally assumed to fail when their degradation level exceeds a threshold level D. Then, for monotonically increasing degradation processes, the unit lifetime X is defined as the operating time to the first, and sole, passage beyond D. In the case of bounded degradation processes, the threshold D is clearly smaller than the upper bound U.

Likewise, the remaining useful life (RUL) X_t of a unit which is unfailed at the operating time t, say $X_t = \max(0, X - t)$, is defined as the further operating time the unit of age t will spend to exceed the threshold level D. Consequently, based on this definition, the RUL X_t of a unit that at t is already failed is equal to 0.

Then, from the conditional pdf (1) of the degradation increment $\Delta W(t, t+\tau)$, the residual reliability, here intended as the conditional probability that the RUL of the unit X_t, given its state $W(t) = w_t$ at the current age t, exceeds the time x $(x \geq 0)$, is given by:

$$R_t(x|w_t) = \Pr\{\Delta W(t,t+x) \leq D - w_t|W(t)=w_t\} = \frac{\gamma\big(\Delta g(w_t,D);\Delta\eta(t,t+x)\big)}{\Gamma\big(\Delta\eta(t,t+x)\big)}, \quad 0 < w_t < D. \quad (11)$$

From the residual reliability (11), replacing $\Delta g(w_t, D)$ and $\Delta\eta(t, t+x)$ by $g(D)$ and $\eta(x)$, the reliability function of a new unit results in:

$$R(x) = \Pr\{W(x) \leq D\} = \frac{\gamma(g(D);\eta(x))}{\Gamma(\eta(x))}. \quad (12)$$

Finally, the mean RUL $E\{X_t|W(t)=w_t\}$ and the mean lifetime $E\{X\}$ of a new unit are:

$$E\{X_t|W(t)=w_t\} = \int_0^\infty R_t(x|w_t)\mathrm{d}x = \int_0^\infty \frac{\gamma(\Delta g(w_t,D);\Delta\eta(t,t+x))}{\Gamma(\Delta\eta(t,t+x))}\mathrm{d}x, \quad (13)$$

$$E\{X\} = \int_0^\infty R(x)\mathrm{d}x = \int_0^\infty \frac{\gamma(g(D);\eta(x))}{\Gamma(\eta(x))}\mathrm{d}x. \quad (14)$$

3. The likelihood function

Let us suppose that m identical units operate under the same conditions and that the unit i $(i = 1,\dots,m)$ is inspected n_i times at the ages $t_{i,j}$ $(j = 1,\dots,n_i)$. Moreover, let $w_{i,j} = W(t_{i,j})$ denote the degradation level of the unit i measured at the inspection time $t_{i,j}$.

From (1), the conditional pdf of $\Delta W(t_{i,j-1}, t_{i,j})$, given $W(t_{i,j-1}) = w_{i,j-1}$, is:

$$f_{\Delta W(t_{i,j-1},t_{i,j})|W(t_{i,j-1})}\big(\delta_{i,j}|w_{i,j-1}\big) = g'\big(w_{i,j-1}+\delta_{i,j}\big)\frac{\big(\Delta g_{i,j}\big)^{\Delta\eta_{i,j}-1}}{\Gamma\big(\Delta\eta_{i,j}\big)}\, e^{-\Delta g_{i,j}}, \quad 0 < \delta_{i,j} < U - w_{i,j-1}, \quad (15)$$

where $\delta_{i,j} = w_{i,j} - w_{i,j-1}$, and (from (5)) $g'(w) = \beta U/(U-w)^2$, $\Delta g_{i,j} = \beta\big[w_{i,j}/(U - w_{i,j}) - w_{i,j-1}/(U - w_{i,j-1})\big]$, and $\Delta\eta_{i,j} = (t_{i,j}/a)^b - (t_{i,j-1}/a)^b$, with $w_{i,0} = t_{i,0} = 0$ for all i. Then, the likelihood function relative to the data $\mathbf{w} = (w_{1,1},\dots,w_{i,n_i},\dots,w_{m,1}, \dots, w_{m,n_i})$ is:

$$L(\mathbf{w}; a, b, \beta, U) = \prod_{i=1}^{m}\prod_{j=1}^{n_i} f_{\Delta W(t_{i,j-1},t_{i,j})|W(t_{i,j-1})}\big(\delta_{i,j}|w_{i,j-1}\big)$$

$$= U^N \prod_{i=1}^{m} \left(\prod_{j=1}^{n_i} \beta^{\Delta\eta_{i,j}} / (U - w_{i,j})^2 \right) \times \frac{\left(\frac{w_{i,j}}{U-w_{i,j}} - \frac{w_{i,j-1}}{U-w_{i,j-1}} \right)^{\Delta\eta_{i,j}-1}}{\Gamma(\Delta\eta_{i,j})} e^{-\frac{\beta w_{i,n_i}}{U-w_{i,n_i}}}, \quad (16)$$

where $N = \sum_{i=1}^{m} n_i$.

4. The Bayesian inferential procedure

A Bayesian inferential approach is here proposed which allows the analyst to incorporate into the inferential procedure some types of prior information on the observed degradation phenomenon that he may possess. Both vague and informative priors are proposed, depending on the amount of prior information that is supposed to be available. In particular, since the upper bound U is a physical characteristic of a bounded degradation process, it seems very reasonable to assume that the analyst may possess prior information on U stemming from past experience with similar degrading units. Obviously, a first information on U is that it must be greater than the maximum observed degradation level w_{M}. Further information on U may be available, depending on the application. In the following, several prior distributions on U are proposed, reflecting different degrees of knowledge.

(i) No information is available on U, except that $U > w_{\mathrm{M}}$. Thus, the improper (vague) decreasing prior is used:

$$\pi(U) = 1/U , \qquad U > w_{\mathrm{M}} . \quad (17)$$

(ii) A lower bound w_{L}, with $w_{\mathrm{L}} > w_{\mathrm{M}}$, can be formulated, and the improper decreasing prior is used:

$$\pi(U) = 1/U , \qquad U > w_{\mathrm{L}} . \quad (18)$$

(iii) An interval $(w_{\mathrm{L}}, w_{\mathrm{U}})$ of equally probable values for U, with $w_{\mathrm{L}} \geq w_{\mathrm{M}}$, can be formulated on U, and hence the uniform prior over the interval $(w_{\mathrm{L}}, w_{\mathrm{U}})$ is used:

$$\pi(U) = \frac{1}{w_{\mathrm{U}} - w_{\mathrm{L}}} , \quad w_{\mathrm{L}} \leq U \leq w_{\mathrm{U}} . \quad (19)$$

(iv) The analyst can formulate a prior mean $E\{U\}$ and variance $V\{U\}$ on U, still under the constraint $U > w_{\mathrm{M}}$. Then, the following 3-parameter gamma distribution, with location parameter equal to w_{M}, is used:

$$\pi(U) = \frac{q^p (U - w_{\mathrm{M}})^{p-1}}{\Gamma(p)} e^{-q(U - w_{\mathrm{M}})} , \qquad U > w_{\mathrm{M}} , \quad (20)$$

where the parameters p and q can be computed as $q = (E\{U\} - w_{\mathrm{M}})/V\{U\}$ and $p = q\,(E\{U\} - w_{\mathrm{M}})$.

In addition, it is plausible to suppose that the analyst possesses information on the presence of an inflection point. This information can be converted into a prior information on b of $\eta(t) = (t/a)^b$ because, as discussed in the Section 2, the mean function has an inflection point when b is larger than 1 (and the larger b, the more evident the inflection is), while it is concave when b is lower than or equal to 1 (and the smaller b, the more marked the initial concavity of the mean function is). Thus, several prior distributions on b are proposed in the following, reflecting different degrees of knowledge on the shape of the mean function $E\{\Delta W(t)\}$.

(i) No information on the shape of $E\{\Delta W(t)\}$ is available, and hence the (improper) vague prior on b is used:

$$\pi(b) = 1/b , \qquad b > 0 , \quad (21)$$

(ii) The only prior information available is that $E\{\Delta W(t)\}$ has no inflection point, and hence the uniform prior is used:

$$\pi(b) = 1 , \quad 0 \leq b \leq 1 , \quad (22)$$

(iii) The analyst knows that the $E\{\Delta W(t)\}$ has no inflection point and is also able to formulate a prior mean $E\{b\}$ and a prior variance $V\{b\}$ on b, with $b \leq 1$. Then, the following beta prior is used:

$$\pi(b) = \frac{b^{r-1}(1-b)^{s-1}}{\mathrm{B}(r,s)} , 0 \leq b \leq 1 , \quad (23)$$

where the parameters r and s can be obtained by $r = E^2\{b\}(1 - E\{b\})/V\{b\} - E\{b\}$ and $s = r/E\{b\} - r$.

(iv) The only available prior information is that $E\{\Delta W(t)\}$ has an inflection point, and hence the (improper) vague prior is used:

$$\pi(b) = 1/b , \qquad b > 1 , \quad (24)$$

(v) The analyst, besides knowing that $E\{\Delta W(t)\}$ has an inflection point, can also formulate prior mean $E\{b\}$ and variance $V\{b\}$ of b, with $b > 1$. Then, the following 3-parameter gamma distribution, with location parameter equal to 1, is used:

$$\pi(b) = \frac{s^r (b-1)^{r-1}}{\Gamma(r)} e^{-s(b-1)} , \qquad b > 1 , \quad (25)$$

where the parameters r and s can be computed as $s = (E\{b\} - 1)/V\{b\}$ and $r = s\,(E\{b\} - 1)$.

Finally, we assume that no information is available on the other two process parameters β and a, and hence the uniform vague prior pdfs:

$$\pi(\beta) = 1/\beta_U \quad \text{and} \quad \pi(a) = 1/a_U \quad (26)$$

over the intervals $(0, \beta_U)$ and $(0, a_U)$ are used for β and a, respectively, where the values of β_U and a_U are sufficiently large.

Assuming the prior independence of $a, b, \beta,$ and U, from the above suggested prior distributions, the joint posterior pdf of the BTGP parameters is:

$$\pi(a, b, \beta, U|\boldsymbol{w}) \propto L(\boldsymbol{w}; a, b, \beta, U)\, \pi(U)\, \pi(b), \quad (27)$$

which can be used to formulate posterior estimates of any process parameter and function thereof.

From (27), the posterior predictive pdf of the increment $\Delta W_i = W(t_{i,n_i} + \tau) - W(t_{i,n_i})$ of unit i over the future time interval $(t_{i,n_i}, t_{i,n_i} + \tau)$, given $W(t_{i,n_i}) = w_{i,n_i}$, is given by:

$$f_{\Delta W_i|\boldsymbol{W}}(\delta|\boldsymbol{w}) = \int_U \int_\beta \int_b \int_a \pi(a, b, \beta, U|\boldsymbol{w}) \times f_{\Delta W_i|W(t_{i,n_i})}(\delta|w_{i,n_i})\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}\beta\, \mathrm{d}U, \quad (28)$$

where $\boldsymbol{W} = (W(t_{1,1}), \dots, W(t_{1,n_1}), \dots, W(t_{m,1}), \dots, W(t_{m,n_m}))$, and from (1):

$$f_{\Delta W_i|W(t_{i,n_i})}(\delta|w_{i,n_i}) = \frac{\beta^{\Delta\eta(t_{i,n_i}, t_{i,n_i}+\tau)}\, U}{\left(U - w_{i,n_i} - \delta\right)^2} \times \frac{\left(\frac{w_{i,n_i}+\delta}{U - w_{i,n_i} - \delta} - \frac{w_{i,n_i}}{U - w_{i,n_i}}\right)^{\Delta\eta(t_{i,n_i}, t_{i,n_i}+\tau)-1}}{\Gamma\left(\Delta\eta(t_{i,n_i}, t_{i,n_i} + \tau)\right)} \times \exp\left\{-\beta\left(\frac{w_{i,n_i} + \delta}{U - w_{i,n_i} - \delta} - \frac{w_{i,n_i}}{U - w_{i,n_i}}\right)\right\} \quad (29)$$

with $\Delta\eta(t_{i,n_i}, t_{i,n_i} + \tau) = \left[(t_{i,n_i} + \tau)/a\right]^b - (t_{i,n_i}/a)^b$.

From (28) and (29), the posterior mean of the (conditional) degradation increment $\Delta W_i|\boldsymbol{W} = \boldsymbol{w}$ of the unit i during the future time interval $(t_{i,n_i}, t_{i,n_i} + \tau)$ is given by:

$$E\{\Delta W_i|\boldsymbol{W} = \boldsymbol{w}\} = \int_U \int_\beta \int_b \int_a \left(\int_0^{U - w_{i,n_i}} \delta\, f_{\Delta W_i|\boldsymbol{W}}(\delta|\boldsymbol{w})\, \mathrm{d}\delta\right) \times \pi(a, b, \beta, U|\boldsymbol{w})\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}\beta\, \mathrm{d}U. \quad (30)$$

5. The Monte Carlo Markov Chain procedure

The Bayesian inferential procedure presented in the Section 4 would require multivariate integrations to be numerically performed. However, this approach is often unfeasible and/or highly time consuming in practice. Therefore, a Monte Carlo Markov Chain (MCMC) method for posterior sampling is proposed to reduce both the computational burden and the execution time of the routines. At this aim, the software package OpenBUGS (Lunn et al. (2009)) is used to implement the MCMC technique based on the adaptive Metropolis algorithm. In particular, we first generate a four-dimensional pseudo-random vector sample of size M, that is $\boldsymbol{\theta}_j = (a_j, b_j, \beta_j, U_j)$ $(j = 1, \dots, M)$, from the joint posterior pdf $\pi(a, b, \beta, U|\boldsymbol{w})$ in (27) after a sufficiently large burn-in period performed to make negligible the influence of the starting point of the numerical procedure.

Hence, from the vector sample $\boldsymbol{\theta}_j$, the posterior mean and variance, and the $(1 - \gamma)$ highest posterior density (HPD) interval of each parameter are estimated. In particular, the posterior mean is given by the mean of the corresponding elements of the posterior sample, for instance

$$E\{a|\boldsymbol{W} = \boldsymbol{w}\} = \int_U \int_\beta \int_b \int_a a\, \pi(a, b, \beta, U|\boldsymbol{w}) \times \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}\beta\, \mathrm{d}U \cong \sum_{j=1}^{M} a_j/M, \quad (31)$$

whereas the $(1 - \gamma)$ HPD interval is obtained by ordering the posterior sample and selecting the shortest interval containing the fraction $(1 - \gamma)$ of the sample.

The posterior sample of any function $h(\boldsymbol{\theta})$ of the BTGP parameters, such as the mean degradation in (7), the residual reliability in (11) or the mean lifetime in (14), is simply given by $h_j = h(\boldsymbol{\theta}_j)$ $(i = 1, \dots, M)$, from which the posterior pdf, the posterior mean, and the HPD interval of $h(\boldsymbol{\theta})$ are easily computed.

Table 1. Wear $w_{i,k}$ [mm] accumulated by liner i up to the inspection time $t_{i,k}$ [h].

i	$t_{i,1}$	$w_{i,1}$	$t_{i,2}$	$w_{i,2}$	$t_{i,3}$	$w_{i,3}$	$t_{i,4}$	$t_{i,4}$
1	11,300	0.90	14,680	1.30	31,270	2.85		
2	11,300	1.50	21,970	2.00				
3	12,300	1.00	16,300	1.35				
4	14,810	1.90	18,700	2.25	28,000	2.75		
5	10,000	1.20	30,450	2.75	37,310	3.05		
6	6,860	0.50	17,200	1.45	24,710	2.15		
7	2,040	0.40	12,580	2.00	16,620	2.35		
8	7,540	0.50	8,840	1.10	9,770	1.15	16,300	2.10

For example, the posterior sample $E_j\{W(t)\}$ ($j = 1, \dots, M$) of the mean $E\{W(t)\}$, given in (7), is:

$$E_j\{W(t)\} = \int_0^{U_j} w \, \frac{\beta_j \, U_j}{(U_j - w)^2} \frac{\left(\frac{\beta_j \, w}{U_j - w}\right)^{(t/a_j)^{b_j}-1}}{\Gamma\left((t/a_j)^{b_j}\right)} \times e^{-\frac{\beta_j \, w}{U_j - w}} \, dw, \quad (32)$$

from which, for example, the posterior mean results in:

$$E\{E\{W(t)\}|\boldsymbol{W} = \boldsymbol{w}\} \cong \frac{1}{M} \sum_{j=1}^{M} E_j\{W(t)\} \, . \quad (33)$$

Likewise, the posterior sample $E_j\{X\}$ ($j = 1, \dots, M$) of the mean lifetime $E\{X\}$ of a new unit given in (14) is computed as:

$$E_j\{X\} = \int_0^{\infty} \frac{\gamma\left(\frac{\beta_j w}{U_j - w}; \left(\frac{x_j}{a_j}\right)^{b_j}\right)}{\Gamma\left(\left(\frac{x_j}{a_j}\right)^{b_j}\right)} dx, \quad (34)$$

from which, for example, the posterior mean $E\{E\{X\}\}$ results in:

$$E\{E\{X\}\} \cong \frac{1}{M} \sum_{j=1}^{M} E_j\{X\} \, . \quad (35)$$

The posterior (predicted) sample $\delta_j(\tau)|w_t$ ($j = 1, \dots, M$) of the (conditional) degradation increment $\Delta W(t, t + \tau)|w_t$ of the BTGP, given $W(t) = w_t$, was obtained by using the conditional pdf (1). In particular, by applying the method of composition (see, e.g., Tanner (1996)), the posterior sample $\delta_j(\tau)|w_t$ ($j = 1, \dots, M$) is obtained by using a two-steps procedure: at first, a pseudo-random sample $z_j(\tau)|w_t$ ($j = 1, \dots, M$) of size M is generated from a gamma distribution with unit scale parameter and shape parameter $\eta_j(t, t + \tau) = [(t + \tau)/a_j]^{b_j} - (t/a_j)^{b_j}$, and then, from (6), each element $z_j(\Delta t)|w_t$ is transformed into the element $\delta_j(\tau)|w_t$ as follows:

$$\delta_j(\tau)|w_t = \frac{U_j}{\frac{\beta_j}{z_j(\tau)|w_t + \beta_j \frac{w_t}{U_j - w_t}} + 1} - w_t \, , \quad (36)$$

Then, from the posterior sample $\delta_j(\tau)|w_t$ ($j = 1, \dots, M$), the posterior pdf, the posterior mean and variance, and the $(1 - \gamma)$ HPD interval of $\Delta W(t, t + \tau)|w_t$ are easily derived.

6. Numerical application

Let us now consider the wear measurements of the liners of an 8-cylinder Sulzer marine engine, given in Table 1. Observed data are depicted in Figure 2, where data pertaining to the same path are connected by lines.

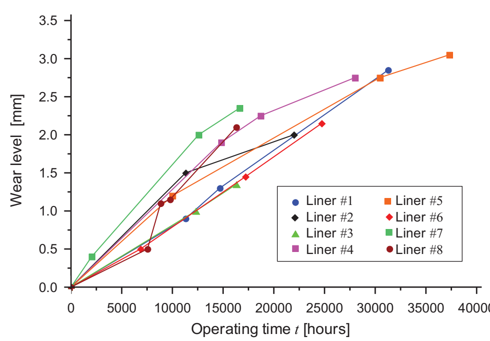

Fig. 2. Observed wear paths of the liners.

Based on a contractual clause, it is assumed that $D = 4$ mm. This wear dataset was previously analyzed in Fouladirad et al. (2023) within the BTGP with $g(w) = \beta[\, w/(U - w)]$ and $\eta(t) =$

$(t/a)^b$ and the maximum likelihood estimates (MLEs) were there obtained.

In order to perform the Bayesian procedure, we have to formulate the prior pdfs for the BTGP parameters. We then assume that the analyst, on the basis of previously observed similar wear phenomena, knows that: *i*) the upper bound U is surely larger that $w_\mathrm{M} = 4.3$ mm, and *ii*) the mean degradation function has an inflection point, so that $b > 1$.

Moreover, from previous experiences, the analyst can also provide the prior values of the mean and variance of U, that is $E\{U\} = 4.6$ mm and $V\{U\} = (0.3 \text{ mm})^2 = 0.09$ mm², and of b, that is $E\{b\} = 1.5$ and $V\{b\} = 0.3^2 = 0.09$. Thus, the analyst chooses:

- the 3-parameter gamma prior (20) on U, with parameters $q = (E\{U\} - w_\mathrm{M})/V\{U\} = 3.333$ and $p = q(E\{U\} - w_\mathrm{M}) = 1.0$, and
- the 3-parameter gamma prior (25) on b, with parameters $s = (E\{b\} - 1)/V\{b\} = 5.556$ and $r = s\,(E\{b\} - 1) = 2.778$.

The prior parameters β_U and α_U are set equal to 200 and 50,000 h, respectively, so as to ensure that these priors are "flat" over the entire region supported by the likelihood. Thus, the joint prior pdf of a, b, β and, U is:

$$g(a,b,\beta,U) \propto \frac{q^p (U - w_\mathrm{M})^{p-1}}{\Gamma(p)} \frac{s^r (b-1)^{r-1}}{\Gamma(r)} \times e^{-q(U-w_\mathrm{M})-s(b-1)},$$

$$a < 50.000\,\text{h}, b > 1, \beta < 200, U > w_\mathrm{M}\,, \quad (37)$$

with $p = 1.0$, $q = 3.333$, $w_\mathrm{M} = 4.3$ mm, $r = 2.778$, and $s = 5.556$.

In order to collect posterior samples of $\boldsymbol{\theta} = (a, b, \beta, U)$ composed by $M = 10^5$ four-dimensional vector elements, we used a burn-in period of 10^5 iterations and a thinning interval equal to 200, guaranteeing very good approximations. The posterior means of a, b, β, and U are $E\{a|\boldsymbol{W} = \boldsymbol{w}\} = 2708$ h, $E\{b|\boldsymbol{W} = \boldsymbol{w}\} = 1.392$, $E\{\beta|\boldsymbol{W} = \boldsymbol{w}\} = 20.28$ and $E\{U|\boldsymbol{W} = \boldsymbol{w}\} = 4.569$ mm, while the corresponding 0.90 HPD intervals are (1509 h, 3820 h), (1.201, 1.568), (9.710, 30.49), and (4.300 mm, 4.907 mm), respectively. For a comparative purpose, the MLEs of a, b, β, and U, given in Fouladirad et al. (2023), are $\hat{a} = 2682$ h, $\hat{b} = 1.434$, $\hat{\beta} = 18.62$ mm, and $\widehat{U} = 4.363$ mm, whereas the approximate 0.90 confidence intervals are (1353 h, 5314 h), (1.119, 1.838), (8.03, 43.13), and (4.3 mm, 5.506 mm), respectively.

Figures 3 and 4 show the posterior means, the empirical estimates, and the 0.90 HPD intervals of $E\{W(t)\}$ and $V\{W(t)\}$.

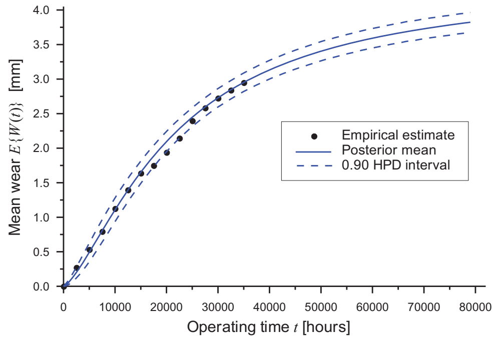

Fig. 3. Posterior mean, empirical estimate, and 0.90 HPD intervals of the mean degradation level.

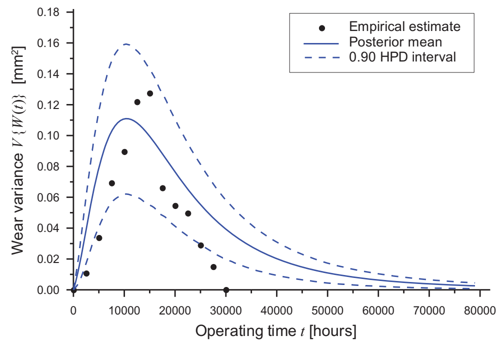

Fig. 4. Posterior mean, empirical estimate, and 0.90 HPD intervals of the variance of the degradation level.

Both the posterior mean of $E\{W(t)\}$ and $V\{W(t)\}$ in Figures 3 and 4 fit very well the corresponding empirical estimates.

Figure 5 shows the estimates of the residual reliability (11) of the liners #3 and 5, given the degradation level w_t=1.35 mm and 3.05 mm at t equal to 16,300 h and 37,310 h, respectively. Liners #3 and 5 have been chosen because they are the liners with the lowest and the highest degradation level, at the last inspection time.

In Table 2, the posterior mean and the 0.90 HPD interval both of the lifetime $E\{X\}$ of a new liner and of the RUL of the liners #1-8 are given. For a comparative purpose, the MLE of $E\{X\}$ is 109,994 h, whereas the MLEs of the mean RUL of the liners #3 and 5 are 95,542 h and 72,836 h, respectively. Thus, the Bayes method provides more pessimistic estimates of $E\{X\}$ and RUL.

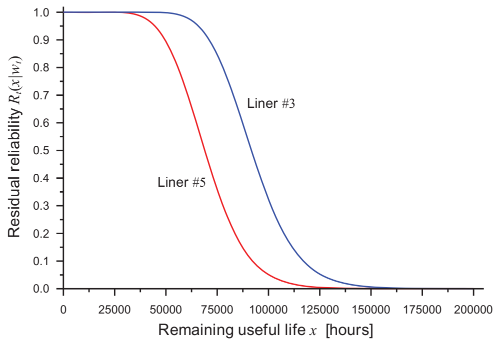

Fig. 5. Posterior mean of the residual reliability $R_t(x|w_t)$ of the liners #3 and 5.

Table 2. Posterior mean and 0.90 HPD interval of the lifetime of a new liner and of the RUL of the liners

Liner	Posterior mean [h]	0.90 HPD interval [h]
New	97,264	(64,448 , 127,390)
#1	65,463	(33,884 , 94,441)
#2	77,215	(44,985 , 107,281)
#3	83,126	(51,043 , 113,753)
#4	67,997	(36,927 , 98,134)
#5	60,211	(29,978 , 88,571)
#6	75,040	(43,270 , 105,370)
#7	77,009	(44,660 , 107,496)
#8	79,136	(46,476 , 109,399)

Finally, Figure 6 shows the posterior predictive pdf (28) of $\Delta W(t_{i,n_i}, t_{i,n_i} + \tau)$ during the future time interval of width $\tau = 10{,}000$ h, relative to the liners #3 and 5.

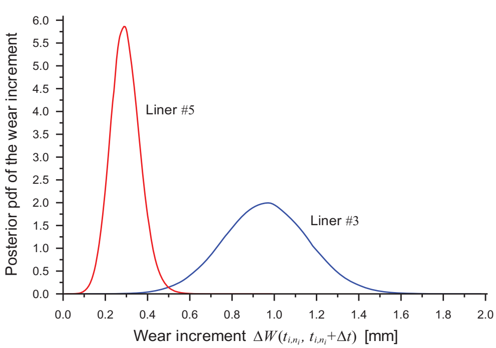

Fig. 6. Posterior pdf of $\Delta W(t_{i,n_i}, t_{i,n_i} + \tau)$ for $\tau = 10{,}000$ h, of liners #3 and 5.

7. Conclusions

In this work, a Bayesian inference procedure for the bounded transformed gamma degradation process (BTGP) has been proposed, when some prior information on the upper bound for the degradation level and on the shape of the mean degradation function are assumed to be available. The use of different prior distributions, modelling different degrees of information on the bounded degradation process under study, has been proposed. Computations have been performed by adopting a Monte Carlo Markov Chain technique.

The posterior distribution of the parameters of the BTGP and of some relevant functions thereof have been derived. From these posterior distributions, the posterior mean and the 0.90 highest posterior density interval have been obtained. Prediction of the degradation increment over a future time interval has been also addressed. The application of the proposed inferential procedure to a set of real wear data of liners of an 8-cylinder marine engine, shows the feasibility of the suggested Bayesian approach.

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