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# An inverse Gaussian process with bathtub-shaped degradation rate function in the presence of random effect and measurement error

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This paper proposes a new inverse Gaussian process with bathtub-shaped degradation rate function, which can account for the joint presence of random effect and measurement error. The distinguishing feature of the proposed model is its ability to describe degradation processes that present a change point, here intended as the point where the bathtub-shaped degradation rate function passes from the decreasing to the increasing phase. Maximum likelihood estimates of the parameters of the model are computed by adopting in a combined manner an expectation-maximization algorithm and a particle filter method. The same particle filter is also used to estimate the mean remaining useful life and to detect the change point. The model is applied to a set of real degradation data of five MOSFETs. Obtained results demonstrate the utility and the affordability of the proposed approach.

*Keywords*: Inverse Gaussian process, random effect, measurement error, Expectation-Maximization, particle filter, bathtub-shaped degradation rate.

## 1. Introduction

The inverse Gaussian process, Wang (2010), has proven to be suited to describe the degradation process of many technological units. Most of the inverse Gaussian processes adopted in the literature have a monotonic (either increasing or decreasing) degradation rate function, here intended as the derivative of the mean function. However, the degradation data of some real-world units, such as the MOSFETs data displayed in Figure 1, exhibit a bathtub-shaped behavior of the degradation rate, with a first phase where the degradation increases with a decreasing rapidity, a second one where it increases with an almost constant rapidity, and a final phase where it increases with increasing rapidity. Classical inverse Gaussian processes adopted in the literature are not able to describe this behavior.
Thus, to overcome this limitation, aiming to fit the MOSFETs data, in this paper we propose a new perturbed inverse Gaussian process with bathtub-shaped degradation rate function, that incorporates a random effect.

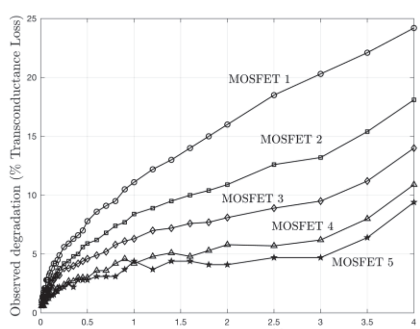


Fig. 1. Degradation paths of the five MOSFETs (Lu et al. 1997)

The need to consider a perturbed model is determined by the circumstance that the paths of the MOSFETs contain negative increments. The random effect captures the unit-to-unit variability shown by the degradation paths of different MOSFETs. The maximum likelihood estimation of the parameters of the proposed model from perturbed data poses severe computational issues. Thus, a maximization procedure that combines an Expectation-Maximization (EM) algorithm and a particle filter method is suggested, which allows to mitigate these issues. The same particle filter is also used to compute the residual reliability of the MOSFETs, their mean remaining useful life, and to detect the change point of their degradation process, which under the proposed model varies randomly from unit-to-unit. The results obtained by analyzing the MOSFETs data demonstrate the utility of the proposed model and the affordability of the suggested estimation procedure.
The paper is structured as follows. Section 2 introduces the proposed model. Sections 3 and 4 focus on likelihood function and remaining useful life. Sections 5 and 6 illustrate the Expectation-Maximization algorithm and the particle filter method. Section 7 reports the results obtained by applying the proposed model to the MOSFETs data and provides details about the preliminary analyses we have conducted to define its structure. Section 8 provides conclusions.

## 2. The perturbed inverse Gaussian process with random effects and bathtub-shaped degradation rate

A perturbed degradation process $\{Z(t); t \geq 0\}$ is customarily defined as:

$$Z(t) = W(t) + \varepsilon(t) \tag{1}$$

where $\{W(t); t \geq 0\}$ is the hidden (also true) degradation process and $\varepsilon(t)$ is a perturbing term, here intended as a measurement error.

It is assumed that, given $W(t_i)$, for any $n > 1$, any $i, j = 1, \dots, n$, and any set of times $t_1, \dots, t_n$, $\varepsilon(t_i)$ and $Z(t_i)$ are conditionally independent of $W(t_j)$, $\varepsilon(t_j)$ and $Z(t_j), \forall i \neq j$ $(i, j = 1, \dots, n)$.
The hidden process $\{W(t); t \geq 0\}$ is assumed to be an inverse Gaussian process with random effect. By following Esposito et al. (2024a), the conditional probability density function (pdf) of the increment $\Delta Y(t, t+\tau) = Y(t+\tau) - Y(t)$, of the inverse Gaussian process $\{Y(t); t \geq 0\}$ is expressed as:

$$f_{\Delta Y(t,t+\tau)}(\delta) = \frac{\Delta\eta(t,t+\tau)}{\sqrt{2\pi\theta^{-1}\delta^3}} e^{-\frac{[\delta-\theta\Delta\eta(t,t+\tau)]^2}{2\theta\delta}}, \tag{2}$$

where $\delta, \tau > 0$, $\Delta\eta(t, t+\tau) = \eta(t+\tau) - \eta(t)$, $\eta(t)$ is a monotonic increasing function, referred to as the age function, and $\theta$ $(\theta > 0)$ is the scale parameter.
The main novelty with respect to Esposito et al. (2024a) is that here, as in Giorgio et al. (2023), Piscopo et al. (2023), and Esposito et al. (2024b), the age function is modeled as:

$$\eta(t) = (t/a_1)^{b_1} + (t/a_2)^{b_2}\,. \tag{3}$$

Therefore, the mean function of $\{Y(t); t \geq 0\}$ is:

$$E\{Y(t)\} = \theta\left(\left(\frac{t}{a_1}\right)^{b_1} + \left(\frac{t}{a_2}\right)^{b_2}\right) \tag{4}$$

and the degradation rate function is:

$$\frac{\mathrm{d}}{\mathrm{d}t}E\{Y(t)\} = \theta\left(\frac{b_1}{a_1}\left(\frac{t}{a_1}\right)^{b_1-1} + \frac{b_2}{a_2}\left(\frac{t}{a_2}\right)^{b_2-1}\right). \tag{5}$$

It is easy to show that when $(b_1 - 1)(b_2 - 1) < 0$ the (5) is bathtub-shaped.
The variance of $Y(t)$ is:

$$V\{Y(t)\} = \theta^2\left(\left(\frac{t}{a_1}\right)^{b_1} + \left(\frac{t}{a_2}\right)^{b_2}\right). \tag{6}$$

Provided that the condition $(b_1 - 1)(b_2 - 1) < 0$ is satisfied, the change point (i.e., the time at which the degradation rate passes from decreasing to increasing) can be computed as:

$$t_{cp} = \left[-\frac{b_2(b_2-1)}{b_1(b_1-1)}\frac{a_1^{b_1}}{a_2^{b_2}}\right]^{1/(b_1-b_2)}.$$

The hidden process $\{W(t); t \geq 0\}$ is constructed by assuming that $b_1$ varies randomly from unit to unit according to the following beta pdf:

$$f_{B_1}(b_1) = \frac{1}{B(s,r)}\, b_1^{s-1}\,(1-b_1)^{r-1}, \tag{7}$$

where $0 < b_1 < 1$ and $r, s > 0$. To emphasize that it is a random variable, hereinafter we will denote $b_1$ by $B_1$ and its realization by $b_1$.
Obviously, under the considered setting, it results $\{W(t); t \geq 0 | B_1 = b_1\} = \{Y(t); t \geq 0\}$.
The conditional cumulative distribution function (cdf) of $\Delta W(t, t+\tau)$ given $B_1 = b_1$ is:

$$F_{\Delta W(t,t+\tau)|B_1}(\delta|b_1) = \Phi\left(\frac{\delta - \theta\Delta\eta(t,t+\tau)}{\sqrt{\theta\delta}}\right) + e^{2\cdot\Delta\eta(t,t+\tau)}\Phi\left(\frac{-\delta - \theta\Delta\eta(t,t+\tau)}{\sqrt{\theta\delta}}\right) \tag{8}$$

By following Esposito et al. (2024a), it is assumed that the error term $\varepsilon(t)$ depends on $W(t)$ and that, given $W(t)=w$, it has the conditional pdf:

$$f_{\varepsilon(t)|W(t)}(\varepsilon|w)=\frac{\left(\alpha(w)\right)^{\beta(w)}e^{-\frac{\alpha(w)}{\varepsilon+w}}}{(\varepsilon+w)^{\beta(w)+1}\Gamma\left(\beta(w)\right)}, \quad (9)$$

where $\varepsilon\geq -w$, $\beta(w)=\varphi w^{2-\nu}+2$, $-\infty<\nu<+\infty$, $\varphi>0$, and $\alpha(w)=w[\beta(w)-1]$.
Moreover, it is assumed that, given $W(t)=w$, $\varepsilon(t)$ and $Z(t)$ do not depend on $B_1$.

Under this setting, from Eqs. (1) and (9), the perturbed measurement $Z(t)$, given $W(t)=w$, is inverse gamma distributed, with conditional pdf:

$$f_{Z(t)|W(t)}(z|w)=\frac{[\alpha(w)]^{\beta(w)}z^{-\beta(w)-1}}{\Gamma\left(\beta(w)\right)}e^{-\frac{\alpha(w)}{z}},$$

conditional mean:

$$E\{Z(t)|W(t)=w\}=w,$$

and conditional variance:

$$V\{Z(t)|W(t)=w\}=\frac{w^{\nu}}{\varphi}.$$

The marginal pdf and cdf of $W(t)$ and $Z(t)$. are not available in closed form. However, by using the law of total mean, the following closed form can be obtained for the marginal mean of $W(t)$:

$$\begin{aligned}E\{W(t)\}&=E\left\{E\{W(t)|B_1\}\right\}\\&=\int_0^1 E\{W(t)|B_1=b_1\}f_{B_1}(b_1)\mathrm{d}b_1\\&=\theta\int_0^1\left(\frac{t}{a_1}\right)^{b_1}\frac{b_1^{s-1}}{B(s,r)}(1-b_1)^{r-1}\mathrm{d}b_1\\&\quad+\theta\left(\frac{t}{a_2}\right)^{b_2}\end{aligned} \quad (10)$$

that, being $E\{Z(t)\}=E\left\{E\{Z(t)|W(t)\}\right\}=E\{W(t)\}$, coincides with the marginal mean of $Z(t)$.
Similarly, the marginal variance of $W(t)$ can be formulated by using the law of total variance:

$$\begin{aligned}V\{W(t)\}&=V\left\{E\{W(t)|B_1\}\right\}+E\left\{V\{W(t)|B_1\}\right\}\\&=\theta^2\int_0^1\left(\frac{t}{a_1}\right)^{2\cdot b_1}f_{B_1}(b_1)\mathrm{d}b_1\\&\quad+\left[2\left(\frac{t}{a_2}\right)^{b_2}+1\right]\theta^2\int_0^1\left(\frac{t}{a_1}\right)^{b_1}f_{b_1}(b_1)\mathrm{d}b_1\\&\quad+\theta^2\left(\frac{t}{a_2}\right)^{2\cdot b_2}+\theta^2\left(\frac{t}{a_2}\right)^{b_2}\\&\quad-\theta^2\left[\int_0^1\left(\left(\frac{t}{a_1}\right)^{b_1}\right)f_{B_1}(b_1)\mathrm{d}b_1+\left(\frac{t}{b_2}\right)^{b_2}\right]^2.\end{aligned} \quad (11)$$

Similarly, the marginal variance of $Z(t)$ is:

$$\begin{aligned}&V\{Z(t)\}\\&\quad=V\left\{E\{Z(t)|W(t)\}\right\}+E\left\{V\{Z(t)|W(t)\}\right\}\\&\quad=V\{W(t)\}+E\left\{\left(W(t)\right)^{\nu}\right\}/\varphi,\end{aligned} \quad (12)$$

where $V\{W(t)\}$ is given in Eq. (11) and the fractional moment:

$$\begin{aligned}E\left\{\left(W(t)\right)^{\nu}\right\}&=E\left\{E\left\{\left(W(t)\right)^{\nu}\middle|B_1\right\}\right\}\\&=\int_0^1\frac{\eta(t)}{\sqrt{2\pi\theta^{-1}}}\left[\int_0^{\infty}w^{\nu-\frac{3}{2}}e^{-\frac{[w-\theta\eta(t)]^2}{2\theta w}}\mathrm{d}w\right]\times\\&\quad\times f_{B_1}(b_1)\mathrm{d}b_1\end{aligned}$$

(in general) should be computed numerically.
It is worth to remark that, both $\{W(t);t\geq 0\}$ and $\{Z(t);t\geq 0\}$ are non-Markovian.

## 3. The likelihood function

Let us assume that the degradation level of $m$ units is measured via inspections which are contaminated by random errors. Moreover, let us denote the vector of model parameters by $\boldsymbol{\xi}=(a_1,a_2,b_2,r,s,\varphi,\nu)$, the age of the $i$th unit ($i=1,\dots,m$) at the $j$th inspection epoch ($j=1,\dots,n_i;n_i\geq 1$) by $t_{i,j}$, the perturbed measurement of its degradation level at $t_{i,j}$ by $Z_{i,j}$, and the realization of $Z_{i,j}$ by $z_{i,j}$.
With these notations, the likelihood function $L(\boldsymbol{\xi};\boldsymbol{z})$ of the perturbed data can be expressed as:

$$L(\boldsymbol{\xi};\boldsymbol{z})=\prod_{i=1}^{m}\prod_{j=1}^{n_i}f_{Z_{i,j}|\boldsymbol{Z}_{i,j-1}}\left(z_{i,j}\middle|\boldsymbol{z}_{i,j-1}\right) \quad (13)$$

where $\boldsymbol{Z}_{i,j}=\left\{Z_{i,1},\dots,Z_{i,j}\right\}$ is the set of all measurements performed on the unit $i$ until $t_{i,j}$, $\boldsymbol{z}_{i,j}=\left\{z_{i,1},\dots,z_{i,j}\right\}$ is its realization, $\boldsymbol{Z}=\left\{\boldsymbol{Z}_{1,n_1},\dots,\boldsymbol{Z}_{i,n_m}\right\}$ is the entire set of perturbed measurements, $\boldsymbol{z}=\left\{\boldsymbol{z}_{1,n_1},\dots,\boldsymbol{z}_{i,n_m}\right\}$ is its realization, $t_{i,0}=0$, $\boldsymbol{Z}_{i.0}$ and $\boldsymbol{z}_{i,0}$ are the empty set and $f_{Z_{i,1}|\boldsymbol{Z}_{i,0}}\left(z_{i,1}\middle|\boldsymbol{z}_{i,0}\right)=f_{Z_{i,1}}\left(z_{i,1}\right)$.
The maximum likelihood estimate (MLE) $\hat{\boldsymbol{\xi}}$ of $\boldsymbol{\xi}$ is defined as the value of $\boldsymbol{\xi}$ that maximizes (over the parameter space) the likelihood function. The likelihood function in Eq. (13) can be efficiently computed by using the particle filter algorithm described in Section 6. However, even by using this tool, the direct maximization of the likelihood is still a very challenging task. For this reason, to retrieve the MLE of $\boldsymbol{\xi}$ we have used, in a combined manner, the EM algorithm described in Section 5 and the particle filter method described

in Section 6.

## 4. The cdf of the RUL

As per the classical failure threshold model, we assume that a unit fails when its true degradation level passes an assigned threshold, say $w_M$. Moreover, we assume that failures are not self-announcing.

Based on these assumptions, the useful life $X$ of a unit is defined as:

$$X = \inf\{x: W(x) > w_M\},$$

and its remaining useful life $RUL(t)$ at the operating time $t$ is defined as:

$$RUL(t) = \max\{0, X - t\}.$$

Accordingly, given that $\{W(t); t \geq 0\}$ is monotonic increasing, we express the conditional cdf of $RUL(t)$, given $\boldsymbol{Z}(t) = \boldsymbol{z}(t)$, as:

$$\begin{aligned} &F_{RUL(t)|\boldsymbol{Z}(t)}(\tau|\boldsymbol{z}(t)) \\ &= P[RUL(t) \leq \tau|\boldsymbol{Z}(t) = \boldsymbol{z}(t)] \\ &= P[W(t+\tau) > w_M|\boldsymbol{Z}(t) = \boldsymbol{z}(t)] \\ &= 1 - F_{W(t+\tau)|\boldsymbol{Z}(t)}(w_M|\boldsymbol{z}(t)) \\ &= 1 - \int_0^1 \int_0^{w_M} F_{\Delta W(t,t+\tau)|B_1}(w_M - w|b_1) \times \\ &\times f_{B_1,W(t)|\boldsymbol{Z}(t)}(b_1, w|\boldsymbol{z}(t)) \cdot \mathrm{d}w \cdot \mathrm{d}b_1 \qquad (14) \end{aligned}$$

where $\boldsymbol{Z}(t) = \{Z(t_j); j \geq 1, t_j \leq t\}$ is the set of measurements gathered until time $t$ and $\boldsymbol{z}(t) = \{z(t_j); j \geq 1, t_j \leq t\}$ is its realization. The conditional joint pdf $f_{B_1,W(t)|\boldsymbol{Z}(t)}(b_1, w|\boldsymbol{z}(t))$ does not allow for a simple mathematical expression but can be efficiently computed via the particle filter described in Section 6.

Note that, given that failures are not self-announcing, and that perturbed measurements alone cannot allow to say with certainty whether a unit is failed or not, in general it results that $F_{RUL(t)|\boldsymbol{Z}(t)}(0|\boldsymbol{z}(t)) > 0$.

From Eq. (14) the conditional mean $MRUL(t|\boldsymbol{z}(t))$ of $RUL(t)$, given $\boldsymbol{Z}(t) = \boldsymbol{z}(t)$, can be formulated as:

$$\begin{aligned} &MRUL(t|\boldsymbol{z}(t)) \\ &= \int_0^\infty \left(1 - F_{RUL(t)|\boldsymbol{Z}(t)}(\tau|\boldsymbol{z}(t))\right) \cdot \mathrm{d}\tau \qquad (15) \end{aligned}$$

Under the considered model setting, $F_{W(t+\tau)|\boldsymbol{Z}(t)}(w_M|\boldsymbol{z}(t))$, $F_{RUL(t)|\boldsymbol{Z}(t)}(\tau|\boldsymbol{z}(t))$, and $MRUL(t|\boldsymbol{z}(t))$ are not available in closed form but can be computed numerically via the particle filter described in Section 6. Finally, it is worth to note that, for $t < t_1$, given that $\boldsymbol{Z}(t)$ is the empty set, being $F_{W(t+\tau)|\boldsymbol{Z}(t)}(w_M|\boldsymbol{z}(t)) = F_{W(t+\tau)}(w_M)$ the Eq. (14) reduces to:

$$\begin{aligned} &F_{RUL(t)|\boldsymbol{Z}(t)}(\tau|\boldsymbol{z}(t)) \\ &\qquad = \int_0^1 F_{\Delta W(t,t+\tau)|B_1}(w_M|b_1) f_{B_1}(b_1)\mathrm{d}b_1. \end{aligned}$$

## 5. The EM algorithm

The MLEs of model parameters are retrieved by using the EM algorithm (Dempster et al. 1974). The procedure consists of two steps: the E-step (expectation step) and the M-step (maximization step), which are iterated until a certain convergence condition is attained.

To apply the algorithm, it is necessary to define observed and missing data. In this paper, the missing data are:

- The values $\boldsymbol{b}_1 = \{b_{1,1}, \dots, b_{1,m}\}$ of the random parameter $\boldsymbol{B}_1 = \{b_{1,1}, \dots, b_{1,m}\}$, and
- The values $\boldsymbol{w} = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$ of the true degradation levels $\boldsymbol{W} = \{\boldsymbol{W}_1, \dots, \boldsymbol{W}_m\}$ of the $m$ units at the measurement times, where $\boldsymbol{W}_i = \{W_{i,1}, \dots, W_{i,n_i}\}$, $\boldsymbol{w}_i = \{w_{i,1}, \dots, w_{i,n_i}\}$, $W_{i,j} = W(t_{i,j})$, and $w_{i,j}$ is its realization.

Obviously, the observed data are the realizations $\boldsymbol{z} = \{\boldsymbol{z}_1, \dots, \boldsymbol{z}_m\}$ of perturbed measurements $\boldsymbol{Z} = \{\boldsymbol{Z}_1, \dots, \boldsymbol{Z}_m\}$, where $\boldsymbol{Z}_i = \{Z_{i,1}, \dots, Z_{i,n_i}\}$ and $\boldsymbol{z}_i = \{z_{i,1}, \dots, z_{i,n_i}\}$.

So stated, the complete likelihood (i.e., of both missing and observed data) can be expressed as:

$$\begin{aligned} L(\boldsymbol{\xi}; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{b}_1) &= \prod_{i=1}^{m} \prod_{i=1}^{n_i} f_{Z_{i,j}|W_{i,j}}(z_{i,j}|w_{i,j}) \\ &\times \left[2\pi\theta^{-1}\Delta w_{i,j}^3\right]^{-1/2} \Delta\eta_{i,j}\, e^{-\frac{\left[\Delta w_{i,j} - \theta\Delta\eta_{i,j}\right]^2}{2\theta\Delta w_{i,j}}} \times \\ &\times \frac{1}{B(s,r)} b_{1,i}^{s-1} (1 - b_{1,i})^{r-1}, \end{aligned}$$

with $Z_{i,j} = Z(t_{i,j})$, $\Delta\eta_{i,j} = \Delta\eta(t_{i,j-1}, t_{i,j})$, $\Delta w_{i,j} = w_{i,j} - w_{i,j-1}$, and $w_{i,0} = 0\ \forall i$.

Accordingly, the complete log-likelihood function $\ell(\cdot\,;\cdot) = \ln L(\cdot\,;\cdot)$ results in:

$$\begin{aligned} \ell(\boldsymbol{\xi}; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{b}_1) &= \ell_E(\nu, \varphi; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{b}_1) \\ &+ \ell_R(r, s; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{b}_1) + \ell_H(\theta, a_1, a_2, b_2; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{b}_1), \end{aligned}$$

where:

$$\ell_E(\nu, \varphi; \boldsymbol{z}, \boldsymbol{w}, \boldsymbol{b}_1) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \beta(w_{i,j}) \ln\left(\alpha(w_{i,j})\right)$$

$$-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{\alpha(w_{i,j})}{z_{i,j}}-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\ln\left(\Gamma\left(\beta(w_{i,j})\right)\right)$$
$$-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\beta(w_{i,j})+1\right)\ln(z_{i,j})$$

is indexed only by the parameters $\nu$ and $\varphi$,

$$\ell_R(r,s;\boldsymbol{z},\boldsymbol{w},\boldsymbol{b}_1)=-m\ln(B(s,r))$$
$$+(s-1)\sum_{i=1}^{m}\log(b_{1,i})$$
$$+(r-1)\sum_{i=1}^{m}\log(1-b_{1,i})$$

is indexed only by the parameters $r$ and $s$, and:

$$\ell_H(\theta,a_1,a_2,b_2;\boldsymbol{z},\boldsymbol{w},\boldsymbol{b}_1)=\sum_{i=1}^{m}\sum_{j=1}^{n_i}\ln(\Delta\eta_{i,j})$$
$$-\frac{n_t}{2}(\ln(2\pi/\theta))-\frac{3}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}\ln(\Delta w_{i,j})$$
$$-\frac{1}{2\theta}\sum_{i=1}^{m}w_{i,n_i}-\frac{\theta}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{(\Delta\eta_{i,j})^2}{\Delta w_{i,j}}+\sum_{i=1}^{m}\eta_{i,n_i},$$

is indexed only by the parameters $\theta$, $a_1$, $a_2$, and $b_2$, with $\eta_{i,n_i}=\sum_{i=1}^{n_i}\Delta\eta_{i,j}=\eta(t_{i,n_i})=(t_{i,n_i}/a_1)^{b_1}+(t_{i,n_i}/a_2)^{b_2}$.

After $h$ iterations, the E-step consists in computing the conditional mean:

$$Q(\boldsymbol{\xi}|\boldsymbol{\xi}^{(h)})=E\{\ell(\boldsymbol{\xi};\boldsymbol{z},\boldsymbol{W},\boldsymbol{B}_1)|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\}$$
$$=Q_E(\nu,\varphi|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)})+Q_R(r,s|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)})$$
$$+Q_H(\theta,a_1,a_2,b_2|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}),$$

where $\boldsymbol{\xi}^{(h)}$ indicates the estimate of $\boldsymbol{\xi}$ obtained by performing the $h$th M-step and the functions $Q_E(\cdot)$, $Q_R(\cdot)$, and $Q_H(\cdot)$ can be expressed as:

$$Q_E(\nu,\varphi|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)})$$
$$=\sum_{i=1}^{m}\sum_{j=1}^{n_i}E\left\{\beta(W_{i,j})\ln\left(\alpha(W_{i,j})\right)\middle|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\right\}$$
$$-\sum_{i=1}^{m}\sum_{j=1}^{n_i}E\{\beta(W_{i,j})+1|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\}\ln(z_{i,j})$$
$$-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{E\{\alpha(W_{i,j})|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\}}{z_{i,j}}$$
$$-\sum_{i=1}^{m}\sum_{j=1}^{n_i}E\left\{\log\left(\Gamma\left(\beta(W_{i,j})\right)\right)\middle|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\right\},$$

$$Q_R(r,s|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)})=-m\ln(B(s,r))$$
$$+(s-1)\sum_{i=1}^{m}E\{\log(B_{1,i})|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\}$$
$$+(r-1)\sum_{i=1}^{m}E\{\log(1-B_{1,i})|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\},$$

and

$$Q_H(\theta,a_1,a_2,b_2|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)})=-\frac{n_t}{2}(\ln(2\pi/\theta))$$
$$+\sum_{i=1}^{m}\sum_{j=1}^{n_i}E\{\ln(\Delta\eta_{i,j})|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\}$$
$$-\frac{3}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}E\{\ln(\Delta W_{i,j})|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\}$$
$$-\frac{1}{2\theta}\sum_{i=1}^{m}E\{W_{i,n_i}|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\}$$
$$-\frac{\theta}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}E\left\{\frac{(\Delta\eta_{i,j})^2}{\Delta W_{i,j}}\middle|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\right\}$$
$$+\sum_{i=1}^{m}E\{\eta_{i,n_i}|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\}, \quad (16)$$

where $\Delta W_{i,j}=W_{i,j}-W_{i,j-1}$ and $W_{i,0}=0\ \forall i$.

In these equations, the presence of $\boldsymbol{\xi}^{(h)}$ on the right side of the conditional bar indicates that the expectations are computed by setting to $\boldsymbol{\xi}^{(h)}$ the parameter vector $\boldsymbol{\xi}$ of the conditional distributions of $\boldsymbol{W}$ and $\boldsymbol{B}_1$ given $\boldsymbol{Z}=\boldsymbol{z}$. All the expectations are computed via the particle filter algorithm illustrated in Section 6.

The M-step consists in maximizing $Q(\boldsymbol{\xi}|\boldsymbol{\xi}^{(h)})$ with respect to $\boldsymbol{\xi}$. The value of $\boldsymbol{\xi}$ that maximizes $Q(\boldsymbol{\xi}|\boldsymbol{\xi}^{(h)})$ (i.e., the new estimate of $\boldsymbol{\xi}$) is denoted by $\boldsymbol{\xi}^{(h+1)}$. In this paper, the iterative procedure is stopped when the absolute relative difference:

$$\left|\frac{\ln\left(L(\boldsymbol{\xi}^{(h+1)};\boldsymbol{z})\right)-\ln\left(L(\boldsymbol{\xi}^{(h)};\boldsymbol{z})\right)}{\ln(L(\boldsymbol{\xi}^{(h)};\boldsymbol{z}))}\right|,$$

where $L(\cdot\,;\cdot)$ is the likelihood function in Eq. (13), drops below an assigned value. The iterative algorithm is initialized by assigning a tentative estimate of $\boldsymbol{\xi}$, say $\boldsymbol{\xi}^{(0)}$.

The maximization of $Q_E(\cdot)$, $Q_R(\cdot)$, and $Q_H(\cdot)$

should be performed numerically. Indeed, the maximization of $Q_H\left(\theta, a_1, a_2, b_2 \middle| \boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)}\right)$ can be further simplified. In fact, it is possible to obtain in explicit form the parameter $\tilde{\theta}$ which maximizes Eq. (16) as a function of $a_1$, $a_2$, and $b_2$ as:

$$\tilde{\theta}(a_1, a_2, b_2) = \frac{n_t + \sqrt{n_t^2 + 4\sum_{i=1}^{m}\sum_{j=1}^{n_i} E\left\{\frac{(\Delta\eta_{i,j})^2}{\Delta W_{i,j}} \middle| \boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{\xi}^{(h)}\right\} \sum_{i=1}^{m} E\left\{W_{i,n_i} \middle| \boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{\xi}^{(h)}\right\}}}{2\sum_{i=1}^{m}\sum_{j=1}^{n_i} E\left\{\frac{(\Delta\eta_{i,j})^2}{\Delta W_{i,j}} \middle| \boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{\xi}^{(h)}\right\}}$$

Thus, the tentative estimates $a_1^{(h+1)}$, $a_2^{(h+1)}$, and $b_2^{(h+1)}$ can be retrieved by maximizing the 3-parameter function $Q_H(\tilde{\theta}(a_1, a_2, b_2), a_1, a_2, b_2 | \boldsymbol{Z} = \boldsymbol{z}, \boldsymbol{\xi}^{(h)})$ that is obtained by replacing $\theta$ in Eq. (16) with $\tilde{\theta}(a_1, a_2, b_2)$. Next, the tentative estimate $\theta^{(h+1)}$ can be obtained as $\tilde{\theta}(a_1^{(h+1)}, a_2^{(h+1)}, b_2^{(h+1)})$.

## 6. The particle filter algorithm

We use the particle filter algorithm (Doucet and Johansen 2011) to generate a sample of size $N$ from the joint distribution of $\boldsymbol{W}_i$ and $\boldsymbol{B}_{1,i}$ given $\boldsymbol{Z}_i = \boldsymbol{z}_i$, given the value of $\boldsymbol{\xi}$.

The method consists of the following two steps, which must be iterated $n_i$ times.

- Step 1 (prediction step), $j$th iteration: for any $k = 1, \dots, N$, set $B_{1,i}$ to ${}_k^{j-1}b_{1,i}$, generate a pseudorandom realization ${}_k\Delta w_{i,j}$ of $\Delta W_{i,j}$, compute ${}_k^{j-1}w_{i,j} = {}_k\Delta w_{i,j} + {}_k^{j-1}w_{i,j-1}$, and append it to the particle vector ${}_k^{j-1}b_{1,i}, {}_k^{j-1}w_{i,1}, \dots, {}_k^{j-1}w_{i,j-1}$ defined at the $(j-1)$th iteration. The output of this prediction step is a set of $N$ vectors:
$$\begin{gathered} {}_1^{j-1}b_{1,i}, {}_1^{j-1}w_{i,1}, \dots, {}_1^{j-1}w_{i,j-1}, {}_1^{j-1}w_{i,j} \\ \vdots \\ {}_N^{j-1}b_{1,i}, {}_N^{j-1}w_{i,1}, \dots, {}_N^{j-1}w_{i,j-1}, {}_N^{j-1}w_{i,j}, \end{gathered}$$
which will be referred to as particles.
- Step 2 (update step), $j$th iteration: for any $k = 1, \dots, N$, compute the importance weight of the $k$th particle as:
$${}_kq_{i,j} = \frac{f_{Z_{i,j}|W_{i,j}}\left(z_{i,j} \middle| {}_k^{j-1}w_{i,j}\right)}{\sum_{k=1}^{N} f_{Z_{i,j}|W_{i,j}}\left(z_{i,j} \middle| {}_k^{j-1}w_{i,j}\right)},$$
resample the particles as per their importance weights and rename the new particles as:
$$\begin{gathered} {}_1^{j}b_{1,i}, {}_1^{j}w_{i,1}, \dots, {}_1^{j}w_{i,j-1}, {}_1^{j}w_{i,j} \\ \vdots \\ {}_N^{j}b_{1,i}, {}_N^{j}w_{i,1}, \dots, {}_N^{j}w_{i,j-1}, {}_N^{j}w_{i,j}. \end{gathered}$$

In the first prediction step (i.e., for $j = 1$), initialize the algorithm by drawing a pseudorandom sample of size $N$ from the joint distribution of $B_{1,i}$ and $W_{i,1}$, denote its elements by $({}_1b_{1,i}, {}_1w_{i,1}), \dots, ({}_Nb_{1,i}, {}_Nw_{i,1})$, and define the particles as:

$$\begin{gathered} {}_1^0b_{1,i}, {}_1^0w_{i,1} = {}_1b_{1,i}, {}_1w_{i,1} \\ \vdots \\ {}_N^0b_{1,i}, {}_N^0w_{i,1} = {}_Nb_{1,i}, {}_Nw_{i,1}. \end{gathered}$$

The particle ${}_k^{j-1}b_{1,i}, {}_k^{j-1}w_{i,1}, \dots, {}_k^{j-1}w_{i,j-1}, {}_k^{j-1}w_{i,j}$ should be intended as a realization of $B_{1,i}$ and $\boldsymbol{W}_{i,j}$ given $\boldsymbol{Z}_{i,j-1} = \boldsymbol{z}_{i,j-1}$, and the particle ${}_k^{j}b_{1,i}, {}_k^{j}w_{i,1}, \dots, {}_k^{j}w_{i,j-1}, {}_k^{j}w_{i,j}$ as a realization of $B_{1,i}$ and $\boldsymbol{W}_{i,j}$ given $\boldsymbol{Z}_{i,j} = \boldsymbol{z}_{i,j}$. The conditional pdfs that are needed to compute the likelihood function in Eq. (13) can be approximated as:

$$f_{Z_{i,j}|\boldsymbol{Z}_{i,j-1}}\left(z_{i,j} \middle| \boldsymbol{z}_{i,j-1}\right) \cong \frac{\sum_{k=1}^{N} f_{Z_{i,j}|W_{i,j}}\left(z_{i,j} \middle| {}_k^{j-1}w_{i,j}\right)}{N}$$

where ${}_k^{j-1}w_{i,j}$ is the last component of the particle ${}_k^{j-1}b_{1,i}, {}_k^{j-1}w_{i,1}, \dots, {}_k^{j-1}w_{i,j-1}, {}_k^{j-1}w_{i,j}$ generated at $j$th prediction step.

Similarly, for example, the conditional mean of a function $g(B_i, \boldsymbol{W}_i)$ of $B_i$ and $\boldsymbol{W}_i$, given $\boldsymbol{Z}_i = \boldsymbol{z}_i$ and $\boldsymbol{\xi}$, can be computed as:

$$E\{g(B_i, \boldsymbol{W}_i) | \boldsymbol{Z}_i = \boldsymbol{z}_i, \boldsymbol{\xi}\} \cong \frac{\sum_{k=1}^{N} g\left({}_k^{n_i}b_{1,i}, {}_k^{n_i}\boldsymbol{w}_i\right)}{N}. \quad (17)$$

This particle filter algorithm is also used to compute the MLE of the cdf of $RUL(t)$, $F_{RUL(t)}(\tau|\boldsymbol{z}(t))$, given in Eq. (14), and the MLE of the $MRUL(t|\boldsymbol{z}(t))$, given in Eq. (15). In particular, by using the notation introduced in Section 4, if $t_l \le t < t_{l+1}$, so that the set $\boldsymbol{Z}(t) = \{Z(t_j); j \ge 1, t_j \le t\}$ contains $l$ perturbed measurements of the degradation level of a certain unit, given a pseudorandom sample of size $N$ from $W(t_l)|\boldsymbol{Z}(t) = \boldsymbol{z}(t)$, say ${}_1^lw_l, \dots, {}_N^lw_l$, the

MLE of $F_{RUL(t)|\boldsymbol{Z}(t)}(\tau|\boldsymbol{z}(t))$ can be computed as:

$$F_{RUL(t)|\boldsymbol{Z}(t)}(\tau|\boldsymbol{z}(t)) \cong 1 - \frac{\sum_{k=1}^{N} F_{W(t+\tau)|W(t_l)}\left(w_M \middle| {}_k^l w_l\right)}{N}.$$

By using this cdf the $MRUL(\tau|\boldsymbol{z}(t))$ can be computed as in (15).
Then, $MRUL(\tau|\boldsymbol{z}(t))$ is obtained as in Eq. (15).

## 7. Numerical example and a preliminary analysis

In this section, we present the results of a preliminary analysis that allows us to justify some of the assumptions we made to formulate the model, followed by an example of application to the MOSFETs data displayed in Figure 1 (the whole dataset is given in Lu et al. 1997).

As a first step, we have performed a comparative study to check whether there are parameters of the basic inverse Gaussian process (i.e., without random effect), hereinafter referred to as M0, that should be assumed to vary from unit-to-unit.

In particular, in order to contain model complexity, we have assumed that only one out of 7 parameters of M0 at most could be unit-specific, therefore formulating 7 competing models. With five units in the dataset, the 7 considered competing models have 11 parameters (6 common and 5 unit-specific).

Table 1 reports the MLE, $\hat{\ell}$, of the log-likelihood, (second row) and the values, AIC, of the Akaike information criterion index (third row), obtained under the considered competing models, where $\text{AIC} = 2k - 2\hat{\ell}$ and $k$ is the number of parameters (Akaike 1974). Each column refers to the model that treats as unit-specific the parameter indicated in the first row.

Under the model M0 the AIC value is 140.5. The MLEs under M0 can be computed as in Esposito et al. (2024a), by modeling $\eta(t)$ as in (3). Table 1 indicates that, according to the AIC, the best model for the MOSFETs data, among the considered ones, is the model that treats $b_1$ as unit-specific, hereinafter referred to as M1.

Table 1. Results of the preliminary analysis.

| $a_1$ | $b_1$ | $a_2$ | $b_2$ | $\theta$ | $\varphi$ | $\nu$ |
|---|---|---|---|---|---|---|
| −7.94 | −0.356 | −63.1 | −62.8 | −31.6 | −44.9 | −48.5 |
| 37.88 | 22.71 | 148.2 | 147.6 | 85.20 | 111.8 | 119 |

Moreover, to confirm that the model M1 outperforms the model M0, we have conducted a likelihood ratio test with the following null, $H_0$, and alternative, $H_1$, hypotheses:

- $H_0$: all the parameters are common (model M0).
- $H_1$: only $b_1$ is unit specific (model M1).

The test statistic $\Lambda = \hat{\ell}_{H_0} - \hat{\ell}_{H_1}$ (where $\hat{\ell}_{H_0} = -63.25$ and $\hat{\ell}_{H_1} = -0.355$ are the MLEs of $\ell$ under $H_0$ and $H_1$, respectively) resulted equal to 127.2 and it is approximately distributed as a chi-square variable with 4 degrees of freedom. The negligible $p$-value associated with the test conclusively leads to reject the null hypothesis and, consequently, to assume that $b_1$ varies from unit to unit.

The MLEs of model parameters, under model M1, are as follows: $\hat{a}_1 = 0.1087$, $\hat{a}_2 = 24999$, $\hat{b}_2 = 9.132$, $\hat{\theta} = 0.0422$, $\hat{\varphi} = 42.89$, $\hat{\nu} = 0.2338$. The MLEs of $b_1$, obtained for the five MOSFETs are $0.4890$, $0.459$, $0.4324$, $0.4012$, and $0.3786$. Indeed, the use of a beta pdf (7) for $b_1$ is also justified by the circumstance that all the MLEs of $b_1$ are safely smaller than 1.

In addition, given that $\hat{b}_2$ is greater than 1, the beta pdf guarantees that the inequality $(b_1 - 1)(b_2 - 1) < 0$ is surely satisfied. This last condition also ensures that the degradation rate function is bathtub-shaped and (thus) that the change point exists for any given $B_1 = b_1$.

By applying the proposed degradation model to the MOSFETs data we have obtained the following MLEs of its parameters: $\hat{a}_1 = 0.1289$, $\hat{a}_2 = 25360$, $\hat{b}_2 = 9.246$, $\hat{\theta} = 0.0461$, $\hat{r} = 86.61$, $\hat{s} = 65.76$, $\hat{\varphi} = 43.43$, $\hat{\nu} = 0.2610$.

Figure 2 shows that the ML estimates of the mean $E\{Z(t)\}$ and standard deviation $\sqrt{V\{Z(t)\}}$ of $Z(t)$ fits very well the corresponding empirical estimates. The same figure also shows that the MLE of the standard deviation of $Z(t)$ and $W(t)$ almost overlap with each other, revealing that in this application the measurement error is very small.

Figure 3 reports the MLEs of the conditional cdf of $RUL(t)$ given $\boldsymbol{Z}(t) = \boldsymbol{z}(t)$, at the last measurement time, $t = 4 \cdot 10^4\ s$. These values have been defined by setting $w_M = 25\%$.

Finally, Table 2 reports, for each MOSFET, the values of $MRUL(t|\boldsymbol{z}(t))$ (second row) and the conditional mean, $E\{T_{cp}|\boldsymbol{Z}(t) = \boldsymbol{z}(t)\}$, of the change point, at the last measurement time $t = 4 \cdot$

$10^4$ $s$. The mean $E\{T_{cp}|\boldsymbol{Z}(t)=\boldsymbol{z}(t)\}$ of $T_{cp}$ is computed as in Eq. (17), by setting:

$$g(B_i,\boldsymbol{W}_i)=g(B_i)=\left[-\frac{b_2(b_2-1)}{B_i(B_i-1)}\frac{a_1^{B_i}}{a_2^{b_2}}\right]^{\frac{1}{B_i-b_2}}.$$

where $T_{cp}$ is denoted by a capital letter to remark that, depending on $B_i$, the change point varies randomly from unit to unit.

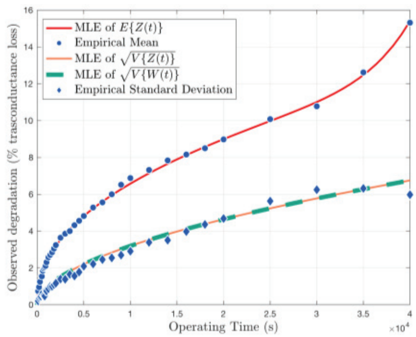


Fig. 2. Empirical and estimated mean and standard deviations of $W(t)$ and $Z(t)$.

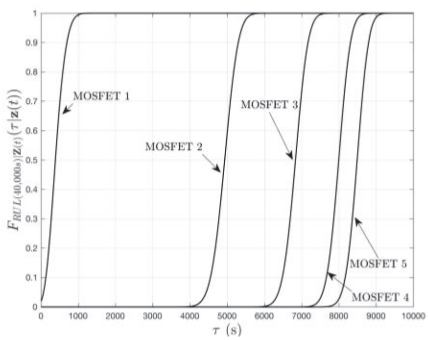


Fig. 3. Conditional cdf of $RUL(t)$ given $\boldsymbol{Z}(t)=\boldsymbol{z}(t)$, at $t=4\cdot10^4$ $s$.

Table 2. Values of $MRUL(t|\boldsymbol{z}(t))$ and $E\{T_{cp}|\boldsymbol{Z}(t)=\boldsymbol{z}(t)\}$ for all MOSFETs in the dataset, at $t=4\cdot10^4$ $s$.

| MOSFET # | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $MRUL$ | 398 | 4923 | 6825 | 7987 | 8482 |
| $t_{cp}$ | 26000 | 25000 | 24000 | 22600 | 22200 |

Fig. 1 and Table 2 give clear evidence of how the $MRUL$ and the mean of the change point vary from unit to unit. Finally, it is worth remarking that, given that the perturbed process is not Markovian, the $MRUL(t|\boldsymbol{z}(t))$ and $E\{T_{cp}|\boldsymbol{Z}(t)=\boldsymbol{z}(t)\}$ reported in Table 2 depend on the entire observed degradation path $\boldsymbol{z}(t)$.

## 8. Conclusions

This paper has proposed a new inverse Gaussian process with bathtub-shaped degradation rate function which can be used in the joint presence of measurement error and random effect. Maximum likelihood estimates have been computed by adopting, in a combined manner, an expectation-maximization algorithm and a particle filter method. The model has been applied to a set of real degradation data of five MOSFETs. Obtained results have demonstrated the utility of the proposed model and the affordability of the suggested estimation approach.

## References

Wang, X., and D. Xu (2010). An inverse Gaussian process model for degradation data. *Technometrics 52(2)*, 188-197.

Lu, J.C., J. Park, and Q. Yang (1997). Statistical inference of a time-to-failure distribution derived from linear degradation data. *Technometrics 39(4)*, 391-400.

Esposito N., A. Mele, B. Castanier, and M. Giorgio (2024a). Misspecification analysis of a gamma- and inverse Gaussian-based perturbed degradation process. *Applied Stochastic models in Business and Industry 40(3)*, 640-667.

Giorgio, M., A. Piscopo, and G. Pulcini (2024). A new Wiener process with bathtub-shaped degradation rate in the presence of random effects. *Applied Stochastic models in Business and Industry 40(3)*, 574-597.

Piscopo, A., N. Esposito, B. Castanier, and M. Giorgio (2023). Remaining useful life estimation of gamma degrading units characterized by a bathtub-shaped degradation rate in the presence of random effect and measurement error. In *proceedings of the 33rd European Safety and Reliability Conference (ESREL2023),* pp. 1258-1265. Research Publishing.

Esposito, N., A. Piscopo, B. Castanier, and M. Giorgio (2024b). An adaptive maintenance policy for gamma degrading units with bathtub-shaped degradation rate function in the presence of random effect. In *proceedings of the 34th European Safety and Reliability Conference (ESREL2024)*, pp. 29-38. Research Publishing.

Dempster A.P., N. M. Laird, and D.B. Rubin (1977). Maximum likelihood from incomplete data via the EM algorithm. *Journal of the royal Statistical Society Series B 39(1)*, 1-2.

Doucet A. and A.M. Johansen (2011). A tutorial on particle filtering and smoothing: fifteen years later. In D. Crisan and B. Rozovskii (Eds.), *The Oxford Handbook of Nonlinear Filtering*, 656-704. Oxford University press.