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An Inverse Gaussian-based Degradation Process with Covariate-Dependent Random Effects

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This paper introduces a new inverse Gaussian process-based degradation model with covariate dependent random effects. The proposed model is suitable for fitting degradation data which cannot be satisfactorily described by treating separately the effect of the covariate and other forms of unit-to-unit variability. The model is applied to degradation data of some integrated circuit devices. Model parameters are estimated by using the maximum likelihood method. To mitigate numerical issues posed by the direct maximization of the likelihood function, the maximum likelihood estimates of the parameters of the model are retrieved by using the expectation-maximization (EM) algorithm. The probability distribution function of the remaining useful life is formulated by using a failure threshold model. Results obtained by applying the model to the considered integrated circuit devices data demonstrate the utility of the proposed model and the affordability of the adopted estimation approach.

Keywords: Covariate, Random Effects, Maximum Likelihood Estimation, Expectation Maximization Algorithm, Remaining Useful Life.

1. Introduction

In this paper we propose a degradation model for the Device B, firstly introduced by Meeker and Escobar (1998).

These data consist of degradation measurements of 34 integrated circuit devices operating at three different levels of an accelerating variable (i.e., a covariate), represented by the junction temperature.

In order to model these data, Meeker et al. (1998) used a path model with two independent random effects and a covariate. Alternatively, Peng (2015) proposed an inverse Gaussian (IG) process with two stochastically dependent random effects and a covariate. Under both these models, the random effects are modeled by using probability density functions (pdfs) that do not depend on the covariate. More recently, Wang et al. (2020) and Wang et al. (2021) proposed, for the Device B data, a homogeneous Wiener process and a homogeneous gamma process, respectively, with a single random effect which depends functionally on the junction temperature.

Inspired by these papers, to model the Device B data, we propose a process, with two stochastically dependent random effects, which in turn functionally depend on the junction temperature that is treated as a covariate.

Under this new model, the temporal variability is described by using an IG process. The random effects are modeled by assuming that one of the parameters follows a gamma distribution, while the other is a deterministic function of the first one. The dependence on the junction temperature is described by using an appropriate link function. This new model is capable of describing degradation phenomena where modeling covariates and random effects separately proves to be inadequate.

Model parameters are estimated by using the maximum likelihood method. The likelihood is not available in closed form and is indexed by many (i.e., 6) parameters. Hence, an expectation-maximization (EM) algorithm is suggested, which allows mitigating the numerical issues posed by its direct maximization.

The rest of the paper is structured as follows. Section 2 presents the proposed degradation process. Section 3 deals with the formulation of cumulative distribution function (Cdf) of the remaining useful life. Section 4 addresses the maximum likelihood (ML) estimation of the parameters of the proposed model. Section 5 illustrates the EM algorithm. Section 6 reports the results of the application of the proposed model to the Device B data. Finally, Section 7 provides some concluding remarks.

2. The Inverse Gaussian Process with Covariate-Dependent Random Effects

The non-homogeneous IG process $\{Y(t), t \geq 0\}$, in its basic form (e.g., see Ye and Chen 2014), is a continuous stochastic process with the following properties: *i*) $Y(0) = 0$ with probability one; *ii*) $Y(t)$ has independent increments; *iii*) for any $t, \tau \geq 0$, the increment $\Delta Y(t, t+\tau) = Y(t+\tau) - Y(t)$ has an IG pdf:

$$f_{\Delta Y(t,t+\tau)}(\delta) = \sqrt{\frac{\lambda\left(\Delta\eta(t,t+\tau)\right)^2}{2\pi \cdot \delta^3}} \times$$

$$\times\, e^{-\frac{\lambda\left(\delta-\Delta\eta(t,t+\tau)\right)^2}{2\delta}}, \quad \delta \geq 0, \lambda > 0, \qquad (1)$$

where $\Delta\eta(t, t+\tau) = \eta(t+\tau) - \eta(t)$ and $\eta(t)$ is an increasing positive function with $\eta(0) = 0$.

The mean $E\{Y(t)\}$ and the variance $V\{Y(t)\}$ of the IG process can be expressed as:

$$E\{Y(t)\} = \eta(t), \qquad (2)$$

and

$$V\{Y(t)\} = \eta(t)/\lambda. \qquad (3)$$

Following Meeker et al. (1998), the age function $\eta(t)$ is modeled by using the bounded function

$$\eta(t) = a[1 - \exp(-bt)]. \qquad (4)$$

Hence, the mean in (2) and the variance in (3) are bounded, yet the process itself remains unbounded.

To account for the presence of heterogeneity among the units we have incorporated into the basic IG process two dependent random effects and a covariate. The random effects are modeled by assuming that the parameters λ and b vary randomly from unit-to-unit. This device allows modelling forms of heterogeneity caused by covariates that are not explicitly included in the model.

Henceforth, to remark that these are random variables, we will denote λ and b by Λ and B and their realizations by λ and b.

The random effects are modeled by assuming that B varies randomly from unit-to-unit according to the gamma pdf:

$$g_B(b) = \frac{c_T{}^d\, b^{d-1}}{\Gamma(d)} e^{-c_T\, b}, \quad d, c_T, b > 0 \qquad (5)$$

and Λ is a deterministic function of B, given by

$$\Lambda = \lambda_0\, B^{-q_2}, \qquad \lambda_0 > 0 \qquad (6)$$

where d and c_T are the shape and scale parameters, respectively, $\Gamma(\cdot)$ is the complete gamma function and $q_2 \in (-\infty, +\infty)$. Notably, when $q_2 = 0$, Λ reduces to the constant value λ_0.

The covariate T (i.e., the junction temperature) is introduced in the model by assuming that the scale parameter c_T of the pdf of B depends on T via the (Arrhenius type) link function (7):

$$c_T = c_0\, e^{q_1/(T+273.15)}, \qquad (7)$$

with $c_0 > 0$ and $q_1 \in (-\infty, +\infty)$. It is worth to remark that, unless $q_2 = 0$, the random variable Λ depends on the covariate T through B.

The link function (7) was already used in Meeker and Escobar (1998) to model the same data. However, in their case, the parameter linked to T was a fixed parameter.

The resulting IG process with random effects and covariates $\{W(t), t \geq 0\}$ is not Markovian.

Moreover, given these modeling assumptions, the following identities hold:

$$\{W(t), t \geq 0 | B = b, \Lambda = \lambda\}$$
$$= \{W(t), t \geq 0 | B = b\}$$
$$= \{Y(t), t \geq 0\}. \qquad (8)$$

where the first equalities easily follow from Eq. (6).

From (8), the conditional pdf and Cdf of the increment $\Delta W(t, t+\tau) = W(t+\tau) - W(t)$, given $B = b$, can be expressed, for any $\delta > 0$, respectively as:

$$f_{\Delta W(t,t+\tau)|B}(\delta|b)$$
$$= \sqrt{\frac{\lambda(b)\left(\Delta\eta(t,t+\tau)\right)^2}{2\pi \cdot \delta^3}}\, e^{-\frac{\lambda(b)\left(\delta-\Delta\eta(t,t+\tau)\right)^2}{2\cdot\delta}} \qquad (9)$$

and

$$F_{\Delta W(t,t+\tau)|B}(\delta|b) = e^{2\,\lambda(b)\,\Delta\eta(t,t+\tau)} \times$$

$$\times \Phi\left(-\sqrt{\frac{\lambda(b)}{\delta}}\,[\delta + \Delta\eta(t, t+\tau)]\right)$$

$$+\Phi\left(\sqrt{\frac{\lambda(b)}{\delta}}\,[\delta - \Delta\eta(t, t+\tau)]\right), \qquad (10)$$

where $\Phi(\cdot)$ is the standard normal Cdf.

From (2) and (3), the conditional mean and the variance of $W(t)$ given $B=b$ are:

$$E\{W(t)|B=b\} = \eta(t), \qquad (11)$$

and

$$V\{W(t)|B=b\} = \eta(t)/\lambda(b). \qquad (12)$$

From (6) and (9), the (marginal) pdf and Cdf of $W(t)$ can be formulated as:

$$f_{W(t)}(w) = \int_0^{+\infty} \sqrt{\frac{\lambda(b)\,(\eta(t))^2}{2\pi\cdot w^3}} \times$$

$$\times \exp\left(-\frac{\lambda(b)\,(\delta - \eta(t))^2}{2\cdot w}\right) g_B(b)\,\mathrm{d}b \qquad (13)$$

and

$$F_{W(t)}(w) = \int_0^w f_{W(t)}(y)\,\mathrm{d}y \qquad (14)$$

respectively, where $g_B(b)$ is the pdf (5) and $\eta(t)$ is a function of b (see Eq. (4)).

Moreover, from (5), (6), (11), and (12), the marginal mean and variance of $W(t)$ can be expressed as:

$$E\{W(t)\} = a\left[1 - \left(\frac{c_T}{c_T + t}\right)^d\right], \qquad (15)$$

and

$$V\{W(t)\}$$

$$= \frac{a}{\lambda_0 c_T^{q_2}} \frac{\Gamma(d+q_2)}{\Gamma(d)} \left(1 - \left(\frac{c_T}{c_T + t}\right)^{d+q_2}\right)$$

$$+ a^2 {c_T}^d \left[\frac{1}{(c_T + 2t)^d} - \frac{{c_T}^d}{(c_T + t)^{2d}}\right]. \qquad (16)$$

The variance $V\{W(t)\}$ exists when $d > -q_2$.

It should be emphasized that under this setting, both the mean and the variance are bounded. In fact, it is:

$$\lim_{t\to+\infty} E\{W(t)\} = a, \qquad (17)$$

and

$$\lim_{t\to+\infty} V\{W(t)\} = \frac{a}{\lambda_0 c_T^{q_2}} \frac{\Gamma(d+q_2)}{\Gamma(d)}. \qquad (18)$$

This implies that, as t goes to infinity, the degradation mean tends the value a, which does not depend on T, and the variance tends to value on the right side of the (18), where c_T depends on T as per Eq. (7).

Let $\boldsymbol{W}(t) = \{W(t_j); j \geq 1, t_j \leq t\}$ and $\boldsymbol{w}(t)$ denote the measurements gathered up to (and included) the time t and their realizations, respectively; the conditional pdf of B given $\boldsymbol{W}(t) = \boldsymbol{w}(t)$ can be expressed as

$$g_{B|\boldsymbol{W}(t)}(b|\boldsymbol{w}(t))$$

$$= \frac{\left[\prod_{j=1}^{l} \eta(t_{j-1}, t_j)\right] b^{d - \frac{l}{2} q_2 - 1}\, e^{-A(b)}}{\int_0^{\infty} \left[\prod_{j=1}^{l} \eta(t_{j-1}, t_j)\right] b^{d - \frac{l}{2} q_2 - 1}\, e^{-A(b)} \mathrm{d}b}, \qquad (19)$$

where l is the number of observations collected up to (and included) t, $\delta_j = w(t_j) - w(t_{j-1})$, and $A(b)$ is given by

$$A(b)$$

$$= \frac{\lambda_0}{2b^{q_2}} \sum_{j=1}^{l} \frac{\left(\delta_j - \Delta\eta(t_{j-1}, t_j)\right)^2}{\delta_j} + c_T b. \qquad (20)$$

3. The Remaining Useful Life

The considered degrading units are assumed to fail when their degradation level passes an assigned failure threshold, say w_M. Accordingly, their useful life X is defined as the first passage time of the degradation process to w_M:

$$X = \inf\{x : \; W(x) > w_M\} \qquad (21)$$

and their remaining useful life (RUL), at the operating time t, say X_t, is defined as

$$X_t = \max\{0, X - t\}. \qquad (22)$$

Thus, from (22), X_t is equal to $X - t$ if the unit at the time t is still unfailed, and it is equal to 0 otherwise.

Considered that the process $\{W(t), t \geq 0\}$ is monotonically increasing and that the conditional process $\{W(t), t \geq 0 | B = b, \Lambda = \lambda\}$ has independent increments, the conditional Cdf of the RUL given $\boldsymbol{W}(t) = \boldsymbol{w}(t)$ can be formulated as:

$$F_{X_t|\boldsymbol{W}(t)}(x|\boldsymbol{w}(t))$$

$$= 1 - F_{W(t+x)|\boldsymbol{W}(t)}(w_M|\boldsymbol{w}(t))$$

$$= 1 - \int_0^{+\infty} \left[F_{\Delta W(t_l, t_l + x)|B}(w_M - w_l|b)\right.$$

$$\left.\times\, g_{B|\boldsymbol{W}(t)}(b|\boldsymbol{w}(t))\right]\,\mathrm{d}b \qquad (23)$$

where $w_l = w(t_l)$, $F_{\Delta W(t, t+\tau)|B}(w_M - w_l|b)$ is the Cdf in Eq. (10) and $g_{B|\boldsymbol{W}(t)}(b|\boldsymbol{w}(t))$ is the pdf in Eqs. (19).

4. The Likelihood Function

Let us assume that the degradation level of m units randomly selected from the heterogeneous population described by the proposed model is measured at selected epochs by mean of ad hoc inspections. Let n_i denote the number of inspections performed on unit i, and let $t_{i,j}, \dots, t_{i,n_i}$ represent the corresponding inspection times. Moreover, let T_i denote the (known) value that the covariate takes in the case of the i–th unit.

Then, the likelihood function related to the i-th unit can be formulated as:

$$\mathcal{L}_i(\boldsymbol{\xi}; \boldsymbol{w_i}) = \left(\frac{\lambda_0}{2\pi}\right)^{\frac{n_i}{2}} \frac{c_T^d}{\Gamma(d)\ \prod_{j=1}^{n_i} \delta_{i,j}^{3/2}}$$

$$\times \int_0^{+\infty} \left[\prod_{j=1}^{n_i} \eta\left(t_{i,j-1}, t_{i,j}\right)\right] b^{d-\frac{n_i}{2}q_2-1} e^{-A_i(b)} \mathrm{d}b \quad (24)$$

where $\boldsymbol{\xi} = \{a, c_0, q_1, d, \lambda_0, q_2\}$ is the vector of model parameters, $\delta_{i,j} = w_{i,j} - w_{i,j-1}$, $w_{i,j}$ is the degradation level of i-th unit observed at the time $t_{i,j}$, $\boldsymbol{w}_i$ is the set $\boldsymbol{w}_i = \{w_{i,1}, \dots, w_{i,n_i}\}$, $\Delta\eta_{i,j}(b) = \eta(t_{i,j}) - \eta(t_{i,j-1})$, $\eta(t_{i,j}) = a[1 - \exp(-b\ t_{i,j})]$, and $A_i(b)$ is defined as

$$A_i(b) = \frac{\lambda_0}{2b^{q_2}} \sum_{j=1}^{n_i} \frac{\left(\delta_{i,j} - \Delta\eta\left(t_{i,j-1}, t_{i,j}\right)\right)^2}{\delta_{i,j}} + c_T b. \quad (25)$$

From (24), the log-likelihood function relative to all m units is given by:

$$\ell(\boldsymbol{\xi}; \boldsymbol{w}) = \sum_{i=1}^{m} \ln\left(\mathcal{L}_i(\boldsymbol{\xi}; \boldsymbol{w_i})\right). \quad (26)$$

The maximum likelihood estimate (MLE) of $\hat{\boldsymbol{\xi}}$ of $\boldsymbol{\xi}$ is the set $\boldsymbol{\xi}$ that maximize the likelihood function over the parameter space.

The direct maximization of the log-likelihood function in Eq. (26) has a complex structure and is indexed by a large number of parameters (i.e., 6). Hence, in order to mitigate the issues posed by its direct maximization, in this paper to retrieve the maximum likelihood estimates we use the EM algorithm, described in Section 5.

5. The EM Algorithm

The EM (Dempster et al. 1977) is an iterative algorithm used for computing the MLEs in the presence of missing and/or incomplete data. It consists of two steps, the expectation (E) step and the maximization (M) step, which are repeated until a convergence criterion is met.

To use the EM is necessary to establish what it is meant by missing and observed data.

In this work, we treat as missing data the unobservable unit-specific values the random parameter $\boldsymbol{B}$ for the considered devices, specifically, the value $\boldsymbol{b} = \{b_1, \dots, b_m\}$ of $\boldsymbol{B}$=$\{B_1, \dots, B_m\}$.

While, obviously, the observed data consist in the realization $\boldsymbol{w} = \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_m\}$ of the set of the measurements $\boldsymbol{W} = \{\boldsymbol{W}_1, \dots, \boldsymbol{W}_m\}$, where $\boldsymbol{W}_i = \{W_{i,1}, \dots, W_{i,m}\}$ and $\boldsymbol{w}_i = \{w_{i,1}, \dots, w_{i,m}\}$.

Under this setting, the complete likelihood function of missing and observed data is:

$$\mathcal{L}_C(\boldsymbol{\xi}; \boldsymbol{w}, \boldsymbol{b}) = \prod_{i=1}^{m} f_{\boldsymbol{W}_i|B_i}(\boldsymbol{w}_i|b_i) \cdot g_{B_i}(b_i) \quad (27)$$

where:

$$f_{\boldsymbol{W}_i|B_i}(\boldsymbol{w}_i|b_i) = \left[\prod_{j=1}^{n_i} \sqrt{\frac{\lambda_0\ \left(\Delta\eta_{i,j}(b_i)\right)^2}{b_i^{q_2} 2\pi\ \delta_{i,j}^3}}\right] \times$$

$$\times \exp\left(-\frac{\lambda_0}{b_i^{q_2}} \sum_{j=1}^{n_i} \frac{\left(\delta_{i,j} - \Delta\eta_{i,j}(b_i)\right)^2}{2\ \delta_{i,j}}\right),$$

and

$$g_{B_i}(b_i) = \frac{c_{T_i}^d\ b_i^{d-1}}{\Gamma(d)} e^{-c_{T_i} b_i},$$

where $\Delta\eta_{i,j}(b_i) = \eta(t_{i,j}) - \eta(t_{i,j-1})$ and $\eta(\cdot)$ (that depends on b_i) is the function in Eq. (4).

Accordingly, the complete log-likelihood function (i.e., $\ell_C(\cdot\,;\cdot) = \ln(\mathcal{L}_C(\cdot\,;\cdot))$) results in:

$$\ell_C(\boldsymbol{\xi}; \boldsymbol{w}, \boldsymbol{b}) = \ell_P(a, \lambda_0, q_2; \boldsymbol{w}, \boldsymbol{b}) + \ell_B(c_0, d, q_1; \boldsymbol{w}, \boldsymbol{b}) \quad (28)$$

where

$$\ell_P(a, \lambda_0, q_2; \boldsymbol{w}, \boldsymbol{b}) = +\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n_i} \ln\left(\frac{\lambda_0\ \left(\Delta\eta_{i,j}(b_i)\right)^2}{b_i^{q_2} 2\pi\ \delta_{i,j}^3}\right) - \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\lambda_0\ \left(\delta_{i,j} - \Delta\eta_{i,j}(b_i)\right)^2}{b_i^{q_2} 2\ \delta_{i,j}}, \quad (29)$$

is indexed only by the parameters a, λ_0 and q_2, and

$$\ell_B(c_0, d, q_1; \boldsymbol{w}, \boldsymbol{b}) = d\ m \cdot \ln(c_0) + d\ q_1 \sum_{i=1}^{m} \frac{1}{273.15 + T_i} + (d-1) \sum_{i=1}^{m} \ln(b_i) - m \ln\left(\Gamma(d)\right) - c_0 \sum_{i=1}^{m} e^{\frac{q_1}{273.15 + T_i}} b_i \quad (30)$$

is indexed only by the parameters c_0, d and q_1.

After h iterations, the $(h+1)$-th E-step consists in computing the conditional expectation $Q(\boldsymbol{\xi}|\boldsymbol{\xi}^{(h)})$ of $\ell_C(\boldsymbol{\xi};\boldsymbol{w},\boldsymbol{b})$ in Eq. (23) with respect $\boldsymbol{B}$, thus

$$Q(\boldsymbol{\xi}|\boldsymbol{\xi}^{(h)}) = E\{\ell_C(\boldsymbol{\xi};\boldsymbol{w},\boldsymbol{B})|\boldsymbol{Z}=\boldsymbol{z},\boldsymbol{\xi}^{(h)}\} = Q_p(a,\lambda_0,q_2|\boldsymbol{W}=\boldsymbol{w},\boldsymbol{\xi}^{(h)}) + Q_B(c_0,d,q_1|\boldsymbol{W}=\boldsymbol{w},\boldsymbol{\xi}^{(h)}) \quad (31)$$

where the set $\boldsymbol{\xi}^{(h)} = \{a^{(h)}, c_0^{(h)}, q_1^{(h)}, d^{(h)}, \lambda_0^{(h)}, q_2^{(h)}\}$ indicates the estimate of $\boldsymbol{\xi}$ performing the h–th M-step, and

$$\begin{aligned} Q_p(a,\lambda_0,q_2|\boldsymbol{W}=\boldsymbol{w},\boldsymbol{\xi}^{(h)}) &= \sum_{i=1}^{m}\frac{n_i}{2}\,E\left\{\ln\left(\frac{\lambda_0}{B_i^{q_2}}\right)\middle|\boldsymbol{W}=\boldsymbol{w},\boldsymbol{\xi}^{(h)}\right\} \\ &+\sum_{i=1}^{m}\sum_{j=1}^{n_i}E\left\{\ln\left(\Delta\eta_{i,j}(B_i)\right)\middle|\boldsymbol{W}=\boldsymbol{w},\boldsymbol{\xi}^{(h)}\right\} \\ &-\sum_{i=1}^{m}\sum_{j=1}^{n_i}E\left\{\frac{\lambda_0\left(\delta_{i,j}-\Delta\eta_{i,j}(B_i)\right)^2}{B_i^{q_2}\,2\,\delta_{i,j}}\middle|\boldsymbol{W}=\boldsymbol{w},\boldsymbol{\xi}^{(h)}\right\} \\ &-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n_i}\ln\left(2\pi\,\delta_{i,j}^3\right), \end{aligned} \quad (32)$$

$$\begin{aligned} Q_B(c_0,d,q_1|\boldsymbol{W}=\boldsymbol{w},\boldsymbol{\xi}^{(h)}) &= -m\ln(\Gamma(d)) \\ &+d\,q_1\sum_{i=1}^{m}\frac{1}{273.15+T_i}+d\,m\ln(c_0) \\ &+(d-1)\sum_{i=1}^{m}E\{\ln(B_i)\,|\boldsymbol{W}=\boldsymbol{w},\boldsymbol{\xi}^{(h)}\} \\ &-c_0\sum_{i=1}^{m}e^{\frac{q_1}{273.15+T_i}}\,E\{B_i|\boldsymbol{W}=\boldsymbol{w},\boldsymbol{\xi}^{(h)}\}. \end{aligned} \quad (33)$$

The presence of $\boldsymbol{\xi}^{(h)}$ on the right side of the conditional bar indicates that the parameters of the conditional distribution of $\boldsymbol{B}$ given $\boldsymbol{W}=\boldsymbol{w}$ used to perform the expectations, are set to $\boldsymbol{\xi}^{(h)}$.
Conditional expectations in Eqs. (32) and (33) can be evaluated by using the following formula:

$$E\{\phi(B_i)|\boldsymbol{W}=\boldsymbol{w},\boldsymbol{\xi}^{(h)}\} = \int_0^{+\infty}\phi(b_i)\,g_{B_i|\boldsymbol{W}_i}(b|\boldsymbol{w}_i,\boldsymbol{\xi}^{(h)})\,\mathrm{d}b_i \quad (34)$$

where $\phi(B_i)$ is any function of B_i and $g_{B_i|\boldsymbol{W}_i}(b|\boldsymbol{w}_i,\boldsymbol{\xi}^{(h)})$ is the function in Eq. (19) with parameters $\boldsymbol{\xi}$ set to $\boldsymbol{\xi}^{(h)}$.
The M-step consists in maximizing the functions in Eq. (31) with respect to $\boldsymbol{\xi}$. The output of this step is denoted by $\boldsymbol{\xi}^{(h+1)}$.
In this paper, the iterative procedure is stopped when the absolute relative difference

$$\left|\frac{\ell(\boldsymbol{\xi}^{(h+1)};\boldsymbol{w}_i)-\ell(\boldsymbol{\xi}^{(h)};\boldsymbol{w}_i)}{\ell(\boldsymbol{\xi}^{(h)};\boldsymbol{w}_i)}\right|,$$

drops below a pre-assigned value, where $\ell(\cdot;\cdot)$ is the log-likelihood function in (26).

6. Example of application

This section reports and discusses the results obtained by applying the proposed model to the Device B data depicted in Fig. 1. These data were firstly given in Meeker and Escobar (1998).
The complete dataset is available at github.com/Auburngrads/SMRD.data. Here we consider the data multiplied by -1, because the original ones are negative and decreasing over time. The dataset consists of measurements of the power drop in the output of $m=34$ integrated circuit devices, operating at three different values of the junction temperature. The power drop is expressed in dB. The temperature T is expressed in degrees Celsius (°C).
The dataset includes:

- 7 devices tested at $T_i = 150, i = 1,..,7$;
- 12 devices tested at $T_i = 195, i = 8,..,19$;
- 15 devices tested at $T_i = 237, i = 20,..,34$.

All devices are inspected at regular time intervals of 150 hours, specifically there are n_i=32 inspections for units operating at $T=150$, n_i=16 inspections at $T=195$, and n_i=8 inspections at $T=237$. Accordingly, the final observation times for the devices are 4000, 2000, and 1000 hours, respectively.

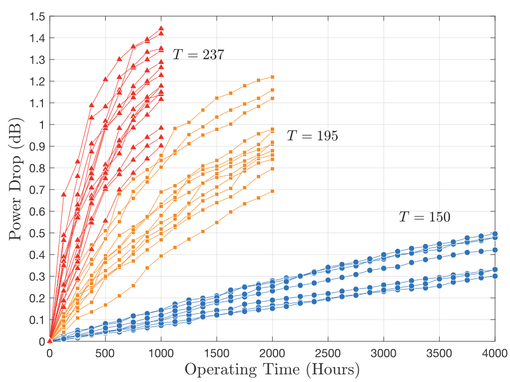

Fig. 1 The degradation paths of 34 Device B and the relative junction temperature.

The MLEs of the parameters are $\hat{a}$=1.432, $\hat{c}_0$=2.491·10^{-3}, $\hat{q}_1$=7.667·10^{-3}, $\hat{d}$=16.54, $\hat{\lambda}_0=0.2738$, and $\hat{q}_2=0.8212$. These MLEs have been retrieved by using the EM algorithm described in Section 5.

The estimated value $\hat{\ell}$ of the log-likelihood is $1.589 \cdot 10^3$. The corresponding value of the Akaike information criterion index (see Akaike 1974) is $AIC = -3.166 \cdot 10^3$. Thus, in particular, this means that, according to the Akaike information criterion, the proposed model fits the device B data better than all the alternative options suggested in Peng (2015), where the best model has an AIC equal to $-2.937 \cdot 10^3$.

Figure 2 shows the empirical estimates of $E\{W(t)\}$, at the three distinct values of the junction temperature, together with the corresponding MLEs of $E\{W(t)\}$, obtained by computing the Eq. (15) at the MLEs of the model parameters with T equal to 150, 195, and 237, respectively.

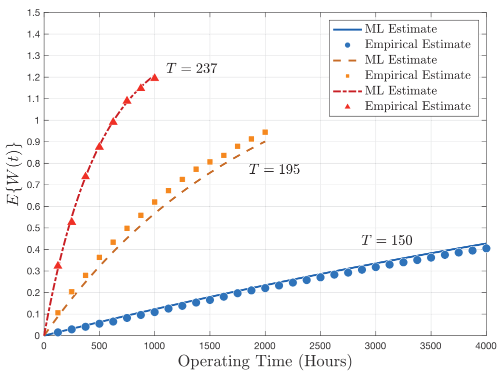

Fig. 2 ML and empirical estimates of the degradation mean $E\{W(t)\}$ for $T = 150, 195$ and 237.

The figure shows that the MLE of the mean function fits adequately the corresponding empirical estimates of the mean at each considered junction temperature.

Figure 3 shows the empirical estimates and the MLEs of the variance $V\{W(t)\}$ at each considered temperature. The MLEs are obtained by computing the Eq. (16) at the MLEs of the model parameters. Again, T is set to 150, 195, and 237, according to the considered set of devices.

It can be observed that the proposed model fits very well the empirical variance at $T = 150$ and provides acceptable results also at $T = 195$ and $T = 237$. Notably, we note that the MLE and the empirical estimate of the variance at $T = 237$ have very similar shapes, both exhibiting a non-monotonic temporal evolution.

Figure 4 reports the 90% probability bands of the degradation process at the three different temperatures (i.e., the MLEs of the 5th and 95th quantiles of $W(t)$). These bands are obtained by using the MLE of the Cdf in (14). We have shaded the area inside the bands for facilitating the interpretation of the figure.

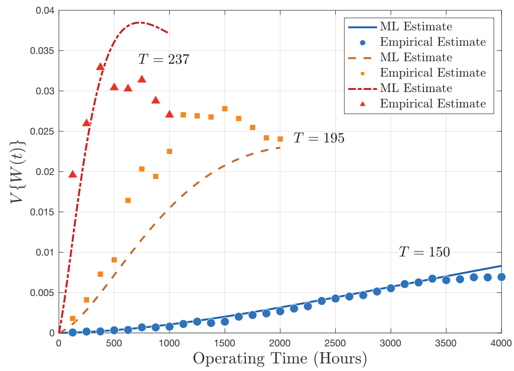

Fig. 3 ML and empirical estimates of the variance $V\{W(t)\}$

The solid blue, orange, and red lines correspond to the probability bands for $T = 150$, 195, and 237, respectively. The dashed lines represent the MLEs of $E\{W(t)\}$ at the corresponding temperatures. The figure shows that the bands include a percentage of observed data close to their nominal value: 95.98% at $T = 150$, 78.12% at $T = 195$, and 90.83% at $T = 237$.

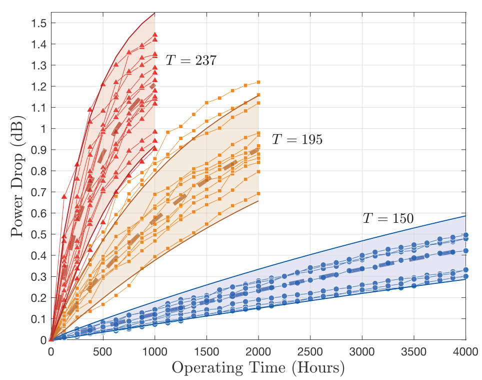

Fig. 4 MLEs of the means and the 90% probability bands at the three different junction temperatures.

By following Meeker and Escobar (1998), we have assumed that the Devices B fail when their degradation level exceeds the threshold value w_M=0.5.

Figure 5 shows the Cdf of the RUL of the devices that operate at T=150, calculated at the last inspection time (t = 4000 hours). The paths of the considered units are represented in Figure 6, where the labels are the same of the ones adopted in Figure 5.

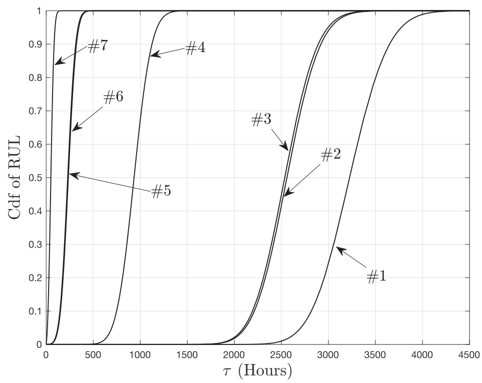

Fig. 5 Cdf of the RUL given all the observed path for Device B units tested at T=150.

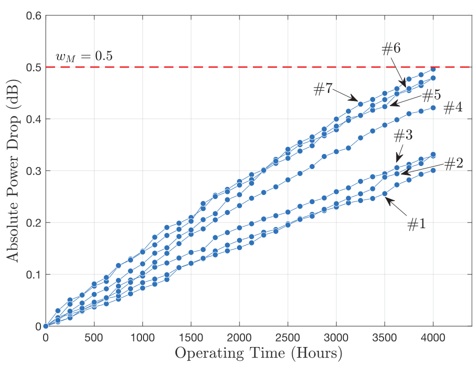

Fig. 6 Degradation paths (in absolute power drop) of Device B at junction temperature T=150.

Cdfs of the RUL are obtained by computing the Eq. (23) at the MLEs of the model parameters. It is noteworthy that the Cdf of the RUL of the units #5 and #6, as well as the Cdfs of the units #2 and #3 are almost overlapped. In fact, given that the degradation process is non-Markovian, this result is obtained because the entire degradation path of the device #5 is very similar to the path of the device #6 and the entire degradation path of the device #2 is very similar to the path of the device #3.

We do not report the corresponding MLEs of the Cdf of the RUL of the devices that operate at T=195 and T=237 because all these devices failed before their last inspection time.

7. Conclusion

In this paper, we have proposed a new inverse Gaussian process-based degradation model that incorporates two stochastically dependent random effects and a covariate. A distinctive characteristic of this model is that the distribution of the random effects depends functionally on the covariate. The main features of the model have been illustrated. The maximum likelihood estimations of the parameters of the model have been addressed. In particular, an expectation-maximization algorithm has been proposed, which allows to easily retrieve the maximum likelihood estimates circumventing the issues posed by the direct maximization of the likelihood function.

The probability distribution function of the remaining useful life has been formulated by using a failure threshold model.

The application of the model to the Device B data demonstrates the utility and the affordability of the proposed approach. Obtained results demonstrated the superiority of the proposed model with respect to other models suggested in the literature for the same degradation data.

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