Proceedings of the 35th European Safety and Reliability & the 33rd Society for Risk Analysis Europe Conference
Edited by Eirik Bjorheim Abrahamsen, Terje Aven, Frederic Bouder, Roger Flage, Marja Ylönen

doi: 10.3850/978-981-94-3281-3_ESREL-SRA-E2025-P0279-cd

Belief Reliability Modeling for Demand-Driven Uncertain Production Systems with Delay Effects

Waichon Lio

Beihang University, Beijing, China. E-mail: liaowj18@buaa.edu.cn

Yang Liu*

*China University of Geosciences, Beijing, China. E-mail: liuy23@buaa.edu.cn, *Corresponding Author*

This paper explores belief reliability modeling for demand-driven uncertain production systems with delay effects. In such systems, demand exhibits cyclical patterns, but real-world complexities introduce epistemic uncertainty, especially with limited data. Delay effects, where initial consumer responses differ significantly from later stages, further complicate demand dynamics. The study introduces a model incorporating delay effects to better predict inventory requirements and system capacity. By defining performance margin and belief reliability based on belief reliability theory, this paper provides a framework to assess system reliability, considering initial inventory, productivity, and demand fluctuations. The proposed model helps producers balance cost efficiency and reliability, ensuring adaptability to both initial demand surges and subsequent stabilization. Additionally, the paper derives analytical expressions for belief reliability and first hitting time of failure, offering practical tools for managing production risks and optimizing system performance under uncertainty. The results highlight the importance of delay effects in shaping demand patterns and system reliability, providing valuable insights for production planning and risk management.

Keywords: Belief reliability, production, uncertainty theory, uncertain renewal process, performance margin, failure, risk analysis

1. Introduction

In the actual production process, when new products are developed or external demand changes dramatically, producers are often trapped by insufficient data and cannot accurately assess the uncertainty within the system. This directly leads to a large cognitive bias when building mathematical models. Under this premise, many scholars have found that classical assumptions such as the law of large numbers and central limit theorem in probability theory are no longer applicable (Liu and Qin (2024), Lio and Liu (2021), Wang and Gao (2025), and Xie and Lio (2024)). To overcome this problem, Liu (2007) founded uncertainty theory in 2007 and Liu (2009) improved it in 2009. Uncertainty theory is a mathematical method based on an axiomatic system, which is designed to analyze epistemic uncertainty in practical systems. To date, uncertainty theory has been applied in many fields, for instance, analyzing uncertain factors in economic activities (Liu and Liu (2022)), prediction of population development trends (Yang and Liu (2024)), financial modeling (Yang and Ke (2023)), and explaining the uncertainty of physical phenomena (Xie et al. (2024)).

As a key branch of uncertainty theory, uncertain renewal process was first proposed by Liu (2008) in 2008, and then in 2010, uncertain renewal reward process was proposed by Liu (2010). Later, Yao and Li (2012) proposed an uncertain alternating renewal process, in which the arrival interval is divided into off-time and on-time, while Zhang et al. (2013) proposed uncertain delayed renewal process. Uncertain renewal process is well suited to describing systems with discontinuous changes in the time dimension, especially in time-varying systems in which discrete behavior often contains complex epistemic uncertainty. Therefore, Liu (2013) first applied uncertain renewal process to the insurance financial system. Subsequently, Yao and Zhou (2018) derived the analytical expression of the distribution of first hitting time under uncertainty theory, and then Liu and Yang

(2020) proposed a more general model for the insurance financial system. For the modeling of queuing system, Yao (2021) obtained the uncertainty distribution of the length of time that the system is continuously busy, and then Liu and Liu (2021) derived the analytical expressions of the distributions of customers' waiting time and system' idle time, respectively. Recently, Liu et al. (2024) verified that the queuing system in practice should be characterized by uncertain renewal process based on actual data of the ride-hailing platform. For the production process, Lio and Liu (2020) first gave the definition of uncertain production risk process, and also obtained some formulas for calculating the shortage index and the distribution of shortage time of the production system. Following that, Lio and Jia (2020) studied a class of uncertain production risk processes with faults and divided them into working and non-working parts. Yu et al. (2024) further used uncertain delayed renewal process to describe production systems.

In reliability engineering practice, the application of uncertain production models relies on the belief reliability theoretical framework built by Kang (2020). For demand-driven production systems focusing on new products, a significant feature is that the initial and subsequent demand processes are adjusted according to the comments of early users, resulting in inconsistent demand intervals. Therefore, it is more appropriate to describe these phenomena by uncertain delayed renewal process. Based on belief reliability theory, this paper gives the definition of the performance margin of demand-driven uncertain production systems with delay effects, and then calculates the belief reliability of the systems. The subsequent contents of this paper are arranged as follows: Section 2 describes the demand-driven uncertain production systems with delay effects, and Section 3 defines the performance margin and draws the formula for calculating the belief reliability of systems. In Section 4, the theorem for calculating the uncertainty distribution of first hitting time of failure is derived. Section 5 gives a brief summary for this paper.

2. Demand-Driven Uncertain Production System

The production system under investigation in this paper is characterized by demands that exhibit cyclical patterns. However, the complexity of real-world environments makes it challenging to precisely estimate the cycle time, as each cycle may exhibit non-negligible difference. This introduces uncertainty in predicting the cycle time for each demand, particularly in scenarios with limited sample sizes, where the uncertainty is predominantly epistemic. This means that the external environment of the production system includes demands influenced by epistemic uncertainty.

In addition to these factors, this paper introduces the concept of "delay effects" into the production system. Delay effects refer to the phenomenon where the initial response to a product, particularly for a new one, differs significantly from the response to subsequent batches. Note that "initial batch" referred here is essentially a broad term encompassing the early stages of product launch. Incorporating delay effects into the production system model allows for a more nuanced understanding of demand dynamics, particularly in the context of new product introductions.

Focusing solely on external demand will cause inventory to decline until it reaches zero, which results in a supply shortage. To address this, the system must engage in production to replenish inventory. Therefore, in a demand-driven uncertain production system with delay effects, cycle times of the demands, arrived demands, the initial inventory level and productivity are critical factors influencing system performance. Specifically, cycle times are assumed to be independent uncertain variables $\tilde{x}_1, \tilde{x}_2, \cdots$. Moreover, $\tilde{x}$ has a different uncertainty distribution from $\tilde{x}_2, \tilde{x}_3, \cdots$ which are identically distributed, i.e., $\tilde{x}_1$ has an uncertainty distribution Φ_1, and $\xi_2, \xi_3, \cdots$ have a common uncertainty distribution Φ. Then the number of demand cycles $\tilde{n}_t$ is the maximal integer n satisfying n demands have reached up to time t, i.e.,

$$\tilde{n}_t = \max_{n \geq 0}\{n \mid \tilde{s}_n \leq t\} \tag{1}$$

where $\tilde{s}_0 = 0$ and $\tilde{s}_n = \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_n$ for

$n \geq 1$. Essentially, the equation (1) is an uncertain delayed renewal process, and Zhang et al. (2013) proved that it is an uncertain process with uncertainty distribution

$$F_{\tilde{n}_t}(x) = 1 - \sup_y \Phi_1(y) \wedge \Phi\left(\frac{t-y}{\lfloor x \rfloor}\right), \quad \forall x \geq 0 \tag{2}$$

where $\lfloor x \rfloor$ is the floor function. Assume also the arrived demands $\tilde{y}_1, \tilde{y}_2, \cdots$ are independent uncertain variables, and $\tilde{y}_1$ has a different uncertainty distribution from $\tilde{y}_2, \tilde{y}_3, \cdots$ which are identically distributed, i.e., the first demand η_1 has an uncertainty distribution Ψ_1, and demands $\eta_2, \eta_3, \cdots$ have a common uncertainty distribution Ψ. Then the total demand up to time t is given by the following definition.

Definition 2.1. Let $\tilde{x}_1, \tilde{x}_2, \cdots$ be iid positive uncertain cycle times, and let $\tilde{y}_1, \tilde{y}_2, \cdots$ be iid positive uncertain demands. Then the delayed renewal reward process

$$\tilde{d}_t = \sum_{i=1}^{\tilde{n}_t} \tilde{y}_i \tag{3}$$

is called an uncertain total demand, where the number of demand cycles $\tilde{n}_t$ is a renewal process with uncertain delayed cycle times $\tilde{x}_1, \tilde{x}_2, \cdots$

Delayed renewal reward process (3) was verified to have an uncertainty distribution

$$\begin{aligned} &F_{\tilde{d}_t}(x) \\ &= (1-\Phi_1(t)) \vee \max_{k\geq 1}\Bigg(1 - \sup_y \Phi_1(y) \\ &\wedge \Phi\left(\frac{t-y}{k}\right)\Bigg) \wedge \left(\sup_y \Psi_1(y) \wedge \Psi\left(\frac{x-y}{k-1}\right)\right). \end{aligned} \tag{4}$$

Here we set $\Psi((x-y)/0) = 1$ for any $x - y \geq 0$.

For initial inventory level a and productivity b, it is obvious that bt is the total production up to time t. Then the surplus inventory level at time t is

$$\tilde{m}_t = a + bt - \tilde{d}_t = a + bt - \sum_{i=1}^{\tilde{n}_t} \tilde{y}_i. \tag{5}$$

For independent cycle times and demands, the uncertainty distribution of equation (5) can be obtained based on the theorem as follows.

Theorem 2.1. *For a demand-driven uncertain production system with delay effects which has the surplus inventory level $\tilde{m}_t = a + bt - \tilde{d}_t$ at time t, where a is the initial inventory level, b is the productivity, and the total demand $\tilde{d}_t$ is a delayed renewal reward process with independent uncertain cycle times $\tilde{x}_1, \tilde{x}_2, \cdots$ and independent uncertain demands $\tilde{y}_1, \tilde{y}_2, \cdots$ Suppose $(\tilde{x}_1, \tilde{x}_2, \cdots)$ and $(\tilde{y}_1, \tilde{y}_2, \cdots)$ are independent uncertain vectors, and the first cycle time $\tilde{x}_1$ has a regular uncertainty distribution Φ_1, cycle times $\tilde{x}_2, \tilde{x}_3, \cdots$ have a common regular uncertainty distribution Φ, while the first demand $\tilde{y}_1$ has a regular uncertainty distribution Ψ_1, and demands $\tilde{y}_2, \tilde{y}_3, \cdots$ have a common regular uncertainty distribution Φ. Then $\tilde{m}_t$ is an uncertain process with uncertainty distribution*

$$\begin{aligned} &F_{\tilde{m}_t}(z) = (1-\Phi_1(t)) \\ &\vee \max_{k\geq 1}\left(1 - \sup_y \Phi_1(y) \wedge \Phi\left(\frac{t-y}{k}\right)\right) \\ &\wedge \left(\sup_y \Psi_1(y) \wedge \Psi\left(\frac{a+bt-z-y}{k-1}\right)\right). \end{aligned} \tag{6}$$

Proof: For a given time t, according to equation (4), the uncertain total demand $\tilde{d}_t$ has an uncertainty distribution

$$\begin{aligned} &F_{\tilde{d}_t}(x) \\ &= (1-\Phi_1(t)) \vee \max_{k\geq 1}\Bigg(1 - \sup_y \Phi_1(y) \\ &\wedge \Phi\left(\frac{t-y}{k}\right)\Bigg) \wedge \left(\sup_y \Psi_1(y) \wedge \Psi\left(\frac{x-y}{k-1}\right)\right). \end{aligned}$$

It follows from operational law in uncertainty theory that $\tilde{m}_t$ is an uncertain variable with uncertainty distribution

$$\begin{aligned} &F_{\tilde{m}_t}(z) \\ &= \sup_{a+bt-x=z} F_{\tilde{d}_t}(x) \\ &= F_{\tilde{d}_t}(a+bt-z) = (1-\Phi_1(t)) \\ &\quad \vee \max_{k\geq 1}\left(1 - \sup_y \Phi_1(y) \wedge \Phi\left(\frac{t-y}{k}\right)\right) \\ &\quad \wedge \left(\sup_y \Psi_1(y) \wedge \Psi\left(\frac{a+bt-z-y}{k-1}\right)\right). \end{aligned}$$

Since time t is arbitrarily chosen, the theorem is then verified.

3. Performance Margin and Belief Reliability

The demand-driven uncertain production system with delay effects is a type of systems of which the function is to meet demands through its supply capacity. However, with the introduction of delay effects, the dynamics of demand become more complex. Specifically, during the initial stages of product launch, demand may surge unpredictably due to consumer novelty and initial reactions, while in later stages, demand stabilizes as consumers base their decisions on product quality and reputation. From this, when the system's initial inventory level is insufficient to meet demand, i.e., when the surplus inventory level drops below 0 (where 0 is a given critical threshold), the production system fails to fulfill its function for customers.

Delay effects cause distinct demand patterns in the initial and mature phases of a product's lifecycle. Thus, it is important to analyze the reliability of the demand-driven uncertain production system with delay effects. Producers aim to maximize profits while ensuring that reliability requirements are met. This necessitates a more precise formula for calculating reliability, taking into account parameters such as initial inventory levels, demand fluctuations, and the impact of delay effects on the demand distribution. By incorporating these factors, producers can better balance cost efficiency and system reliability, ensuring that the production system remains responsive to both the initial surge and the eventual stabilization of demand.

To address the reliability issues of the complicated cases such as the demand-driven uncertain production systems with delay effects, belief reliability theory, which was founded by Kang (2020) through combining probability theory and uncertainty theory to respectively deal with the aleatory uncertainty and epistemic uncertainty, has to be applied. In belief reliability theory, the core concept about performance margin is essentially the adequacy of the system function. Since the function of the demand-driven uncertain production system with delay effects is quantified by the supply capacity, the surplus inventory level can reflect the system adequacy and the following definition is thus given.

Definition 3.1. Assume a demand-driven uncertain production system with delay effects has an initial inventory level a, productivity b and total demand $\tilde{d}_t$. Then the performance margin of the system is defined as

$$\tilde{m}_t = a + bt - \tilde{d}_t. \tag{7}$$

The surplus inventory level in a production system is a larger-the-better (LTB) parameter, with higher levels indicating greater reliability, while demand exceeding supply at any time t signifies unreliability. When considering delay effects, the initial demand surge during product launch adds complexity to maintaining reliability. According to belief reliability theory, the belief reliability for such system should be defined as the belief degree (uncertain measure) that the performance margin remains positive, and its definition is provided as follows.

Definition 3.2. Suppose a demand-driven uncertain production system with delay effects is with the performance margin $\tilde{m}_t = a + bt - \tilde{d}_t$. Then its belief reliability is

$$R_B(t) = \mathcal{M}\{\tilde{m}_t > 0\} = \mathcal{M}\left\{a + bt - \tilde{d}_t > 0\right\}. \tag{8}$$

To simply calculate the belief reliability for the demand-driven uncertain production system with delay effects, the following theorem can be applied due to the analytical solutions of the uncertainty distributions for uncertain renewal process and renewal reward process.

Theorem 3.1. *For a demand-driven uncertain production system with delay effects which has the performance margin $\tilde{m}_t = a + bt - \tilde{d}_t$ at time t, where a is the initial inventory level, b is the productivity, and the total demand $\tilde{d}_t$ is a delayed renewal reward process with independent uncertain cycle times $\tilde{x}_1, \tilde{x}_2, \cdots$ and independent uncertain demands $\tilde{y}_1, \tilde{y}_2, \cdots$ Suppose $(\tilde{x}_1, \tilde{x}_2, \cdots)$*

and $(\tilde{y}_1, \tilde{y}_2, \cdots)$ *are independent uncertain vectors, and the first cycle time* $\tilde{x}_1$ *has a regular uncertainty distribution* Φ_1*, cycle times* $\tilde{x}_2, \tilde{x}_3, \cdots$ *have a common regular uncertainty distribution* Φ*, while the first demand* $\tilde{y}_1$ *has a regular uncertainty distribution* Ψ_1*, and demands* $\tilde{y}_2, \tilde{y}_3, \cdots$ *have a common regular uncertainty distribution* Φ*. Then its belief reliability* R_B *at time* t *is*

$$\begin{aligned} R_B(t) = \Phi_1(t) \wedge \min_{k\geq 1} \Bigg(& \inf_y \Phi_1(y) \\ & \vee \Phi\left(\frac{t-y}{k}\right)\Bigg) \vee \Bigg(1 - \inf_y \Psi_1(y) \\ & \vee \Psi\left(\frac{a+bt-y}{k-1}\right)\Bigg). \end{aligned} \tag{9}$$

Proof: By Theorem 2.1, the performance margin $\tilde{m}_t$ has an uncertainty distribution

$$\begin{aligned} & F_{\tilde{m}_t}(x) = (1-\Phi_1(t)) \\ & \vee \max_{k\geq 1} \left(1 - \sup_y \Phi_1(y) \wedge \Phi\left(\frac{t-y}{k}\right)\right) \\ & \wedge \left(\sup_y \Psi_1(y) \wedge \Psi\left(\frac{a+bt-z-y}{k-1}\right)\right). \end{aligned}$$

Then it follows by Definition 3.2 that we have

$$\begin{aligned} R_B(t) &= \mathcal{M}\{\tilde{m}_t > 0\} \\ &= 1 - \mathcal{M}\{\tilde{m}_t \leq 0\} \\ &= 1 - F_{\tilde{m}_t}(0) \\ &= 1 - \Bigg((1-\Phi_1(t)) \\ &\quad \vee \max_{k\geq 1} \left(1 - \sup_y \Phi_1(y) \wedge \Phi\left(\frac{t-y}{k}\right)\right) \\ &\quad \wedge \left(\sup_y \Psi_1(y) \wedge \Psi\left(\frac{a+bt-y}{k-1}\right)\right)\Bigg) \\ &= \Phi_1(t) \wedge \min_{k\geq 1} \Bigg(\inf_y \Phi_1(y) \\ &\quad \vee \Phi\left(\frac{t-y}{k}\right)\Bigg) \vee \Bigg(1 - \inf_y \Psi_1(y) \\ &\quad \vee \Psi\left(\frac{a+bt-y}{k-1}\right)\Bigg). \end{aligned}$$

The theorem is thus proved.

4. First Hitting Time and Failure

Reliability addresses system failure, which occurs when a system cannot meet its intended function. That means, system failure often involves human factors. In demand-driven production systems, failure appears as supply shortages, which are inevitable over time. Delay effects, such as initial demand surges during product launches, further increase the risk of shortages. Therefore, producers focus on predicting the first hitting time, that is, the moment when failure first occurs, as it can significantly disrupt operations in demand-driven production systems with delay effects. By analyzing this using uncertainty theory, a definition for the first hitting time of failure is given as follows.

Definition 4.1. Suppose $\tilde{m}_t$ is the performance margin of a demand-driven uncertain production system with delay effects. Then the first hitting time τ of failure is

$$\tau = \inf\{t \geq 0 \mid \tilde{m}_t < 0\}. \tag{10}$$

it is clear from Definition 4.1 that the first hitting time of failure is an uncertain variable in the sense of uncertainty theory. In demand-driven uncertain production systems with delay effects, the concept of failure represents the epistemic uncertainty combining the uncertainty of the system itself and those uncertainties arising from delay effects. For a specified time T_F provided as the requirement of system reliability, the belief degree (uncertain measure) of the event that the system experiences failure before T_F can be used to quantify the system failure, i.e.,

$$\text{Failure} = \mathcal{M}\{\tau \leq T_F\}. \tag{11}$$

The following theorem is derived for calculating the result of equation (11).

Theorem 4.1. *For a demand-driven uncertain production system with delay effects which has the performance margin* $\tilde{m}_t = a + bt - \tilde{d}_t$ *at time* t*, where* a *is the initial inventory level,* b *is the productivity, and the total demand* $\tilde{d}_t$ *is a delayed renewal reward process with independent uncertain cycle times* $\tilde{x}_1, \tilde{x}_2, \cdots$ *and independent uncertain demands* $\tilde{y}_1, \tilde{y}_2, \cdots$ *Suppose* $(\tilde{x}_1, \tilde{x}_2, \cdots)$ *and* $(\tilde{y}_1, \tilde{y}_2, \cdots)$ *are independent uncertain vectors, and the first cycle time* $\tilde{x}_1$ *has a regular uncertainty distribution* Φ_1*, cycle times* $\tilde{x}_2, \tilde{x}_3, \cdots$

have a common regular uncertainty distribution Φ, while the first demand $\tilde{y}_1$ has a regular uncertainty distribution Ψ_1, and demands $\tilde{y}_2, \tilde{y}_3, \cdots$ have a common regular uncertainty distribution Φ. Then for the required time T_F, the uncertain measure of system failure can be obtained by

$$\begin{aligned} \text{Failure} = \max_{k\geq 1} & \sup_{x_1+x_2\leq T_F} \Phi_1(x_1) \\ & \wedge \Phi\left(\frac{x_2}{k-1}\right) \wedge (1-\Psi_1(x_3)) \\ & \wedge \left(1-\Psi\left(\frac{a+b(x_1+x_2)-x_3}{k-1}\right)\right). \end{aligned} \tag{12}$$

Here we set

$$\begin{aligned} & \Psi\left(\frac{a+b(x_1+x_2)-x_3}{0}\right) \\ & = \begin{cases} 0, & \text{if } a+b(x_1+x_2)-x_3<0 \\ 1, & \text{if } a+b(x_1+x_2)-x_3\geq 0. \end{cases} \end{aligned} \tag{13}$$

Proof: For any interger k, nonnegative real numbers t and x, let us define

$$\begin{aligned} \alpha_k = & \sup_{x_1+x_2\leq t} \Phi_1(x_1) \\ & \wedge \Phi\left(\frac{x_2}{k-1}\right) \wedge (1-\Psi_1(x_3)) \\ & \wedge \left(1-\Psi\left(\frac{a+b(x_1+x_2)-x_3}{k-1}\right)\right). \end{aligned} \tag{14}$$

Note that

$$\Phi_1(x_1) \quad \text{and} \quad \Phi\left(\frac{x_2}{k-1}\right)$$

are increasing with respect to x_1 and x_2, respectively, $1-\Psi_1(x_3)$ is decreasing with respect to x_3, while

$$1-\Psi\left(\frac{a+b(x_1+x_2)-x_3}{k-1}\right)$$

is decreasing with respect to x_1, x_2 and increasing with respect to x_3. Now we prove that one of the following two alternatives holds:

$$\begin{aligned} & \Phi_1^{-1}(\alpha_k)+(k-1)\Phi^{-1}(\alpha_k)=t, \\ & a+b(\Phi_1^{-1}(\alpha_k)+(k-1)\Phi^{-1}(\alpha_k)) \\ & \quad -\Psi_1^{-1}(1-\alpha_k)-(k-1)\Psi^{-1}(1-\alpha_k)\leq 0; \end{aligned} \tag{15}$$

$$\begin{aligned} & \Phi_1^{-1}(\alpha_k)+(k-1)\Phi^{-1}(\alpha_k)<t, \\ & a+b(\Phi_1^{-1}(\alpha_k)+(k-1)\Phi^{-1}(\alpha_k)) \\ & \quad -\Psi_1^{-1}(1-\alpha_k)-(k-1)\Psi^{-1}(1-\alpha_k)=0. \end{aligned} \tag{16}$$

Assume (x_1^*, x_2^*, x_3^*) is the supremum solution of equation (14). For $k=1$, it is obvious that

$$x_2^*=0, \quad x_3^*=a+b(x_1^*+x_2^*)=a+bx_1^*$$

and

$$\alpha_1=\Phi_1(x_1^*)\wedge(1-\Psi_1(a+bx_1^*)).$$

If $x_1^*+x_2^*=t$, i.e., $x_1^*=t$, then if follows from the assumption that

$$\begin{aligned} \alpha_1 &= \Phi_1(x_1^*)=\Phi_1(t), \\ \alpha_1 &\leq 1-\Psi_1(a+bx_1^*). \end{aligned}$$

Therefore, equation (15) holds; If $x_1^*<t$, then we have

$$\alpha_1=\Phi_1(x_1^*)=(1-\Psi_1(a+bx_1^*)),$$

i.e.,

$$\Phi_1^{-1}(1-\alpha_1)=x_1^*, \quad \Psi_1^{-1}(1-\alpha_1)=a+bx_1^*.$$

Therefore, equation (16) holds. Next, for $k\geq 2$, if $x_1^*+x_2^*=t$, then we have

$$\begin{aligned} & \alpha_k=\Phi_1(x_1^*)\wedge\Phi\left(\frac{x_2^*}{k-1}\right), \\ & \alpha_k\leq(1-\Psi_1(x_3^*))\wedge\left(1-\Psi\left(\frac{a+bt-x_3^*}{k-1}\right)\right). \end{aligned}$$

Then we get

$$\Phi_1^{-1}(\alpha_k)+(k-1)\Phi^{-1}(\alpha_k)=t,$$

and

$$a+bt\leq\Psi_1^{-1}(1-\alpha_k)+(k-1)\Psi^{-1}(1-\alpha_k).$$

It can be summarized that equation (15) holds; If $x_1^*+x_2^*<t$, then we have

$$\begin{aligned} \alpha_k &= \Phi_1(x_1^*)=\Phi\left(\frac{x_2^*}{k-1}\right) \\ &= 1-\Psi_1(x_3^*) \\ &= 1-\Psi\left(\frac{a+b(x_1^*+x_2^*)-x_3^*}{k-1}\right), \end{aligned}$$

i.e.,

$$x_1^* = \Phi_1^{-1}(\alpha_k), \quad \frac{x_2^*}{k-1} = \Phi^{-1}(\alpha_k),$$
$$x_3^* = \Psi_1^{-1}(1-\alpha_k),$$
$$\Psi^{-1}(1-\alpha_k) = \frac{a + b(x_1^* + x_2^*) - x_3^*}{k-1}.$$

Therefore, equation (16) holds. That means one of the alternatives (15) and (16) holds. Next, let us turn attention to uncertainty distribution $\Upsilon(t)$ of the first hitting time τ of failure. On the one hand, it is clear that for each t, we have $\tau \leq t$ if

$$\inf_{0\leq s\leq t} \tilde{m}_s < 0.$$

By equations (15) and (16), we have

$$\begin{aligned}
&\Upsilon(t)\\
&= \mathcal{M}\{\tau \leq t\} \geq \mathcal{M}\left\{\inf_{0\leq s\leq t} \tilde{m}_s < 0\right\}\\
&= \mathcal{M}\Bigg\{ \bigcup_{k=1}^{\infty} \Bigg(\sum_{i=1}^{k} \tilde{x}_i < t,\\
&\qquad a + b\sum_{i=1}^{k} \tilde{x}_i - \sum_{i=1}^{k} \tilde{y}_i < 0 \Bigg)\Bigg\}\\
&\geq \mathcal{M}\Bigg\{ \bigcup_{k=1}^{\infty} (\tilde{x}_1 < \Phi_1^{-1}(\alpha_k)) \cap (\tilde{y}_1 > \Psi_1^{-1}(1-\alpha_k))\\
&\qquad \cap \left(\bigcap_{i=2}^{k} (\tilde{x}_i < \Phi^{-1}(\alpha_k)) \right)\\
&\qquad \cap \left(\bigcap_{i=2}^{k} (\tilde{y}_i > \Psi^{-1}(1-\alpha_k)) \right)\Bigg\}\\
&\geq \bigvee_{k=1}^{\infty} \Bigg(\mathcal{M}\{\tilde{x}_1 < \Phi_1^{-1}(\alpha_k)\} \wedge \mathcal{M}\{\tilde{y}_1 > \Psi_1^{-1}(1-\alpha_k)\}\\
&\qquad \wedge \left(\bigwedge_{i=2}^{k} \mathcal{M}\{\tilde{x}_i < \Phi^{-1}(\alpha_k)\} \right)\\
&\qquad \wedge \left(\bigwedge_{i=2}^{k} \mathcal{M}\left\{\tilde{y}_i > \Psi^{-1}(1-\alpha_k)\right\} \right)\Bigg)\\
&= \bigvee_{k=1}^{\infty} \left(\left(\bigwedge_{i=1}^{k} \alpha_k \right) \wedge \left(\bigwedge_{i=1}^{k} \alpha_k \right)\right)\\
&= \bigvee_{k=1}^{\infty} \alpha_k.
\end{aligned}$$

On the other hand, by equations (15), (16) and Fubini theorem, we have

$$\begin{aligned}
&\Upsilon(t)\\
&= \mathcal{M}\{\tau \leq t\} \leq \mathcal{M}\left\{\inf_{0\leq s\leq t} \tilde{m}_s \leq 0\right\}\\
&= \mathcal{M}\Bigg\{ \bigcup_{k=1}^{\infty} \Bigg(\sum_{i=1}^{k} \tilde{x}_i \leq t,\\
&\qquad a + b\sum_{i=1}^{k} \tilde{x}_i - \sum_{i=1}^{k} \tilde{y}_i \leq 0 \Bigg)\Bigg\}\\
&\leq \mathcal{M}\Bigg\{ \bigcup_{k=1}^{\infty} (\tilde{x}_1 \leq \Phi_1^{-1}(\alpha_k)) \cup (\tilde{y}_1 \geq \Psi_1^{-1}(1-\alpha_k))\\
&\qquad \cup \left(\bigcup_{i=2}^{k} (\tilde{x}_i \leq \Phi^{-1}(\alpha_k)) \right)\\
&\qquad \cup \left(\bigcup_{i=2}^{k} (\tilde{y}_i \geq \Psi^{-1}(1-\alpha_k)) \right)\Bigg\}\\
&\leq \mathcal{M}\Bigg\{ (\tilde{x}_1 \leq \bigvee_{k=1}^{\infty} \Phi_1^{-1}(\alpha_k))\\
&\qquad \cup \bigcup_{i=2}^{\infty} \left(\tilde{x}_i \leq \bigvee_{k=i}^{\infty} \Phi^{-1}(\alpha_k) \right)\\
&\qquad \cup (\tilde{y}_1 \geq \bigwedge_{k=1}^{\infty} \Psi_1^{-1}(1-\alpha_k))\\
&\qquad \cup \bigcup_{i=2}^{\infty} \left(\tilde{y}_i \geq \bigwedge_{k=i}^{\infty} \Psi^{-1}(1-\alpha_k) \right)\Bigg\}\\
&= \mathcal{M}\left\{ \tilde{x}_1 \leq \bigvee_{k=1}^{\infty} \Phi_1^{-1}(\alpha_k) \right\}\\
&\qquad \vee \bigvee_{i=2}^{\infty} \mathcal{M}\left\{ \tilde{x}_i \leq \bigvee_{k=i}^{\infty} \Phi^{-1}(\alpha_k) \right\}\\
&\qquad \vee \mathcal{M}\left\{ \tilde{y}_1 \geq \bigwedge_{k=1}^{\infty} \Psi_1^{-1}(1-\alpha_k) \right\}\\
&\qquad \vee \bigvee_{i=2}^{\infty} \mathcal{M}\left\{ \tilde{y}_i \geq \bigwedge_{k=i}^{\infty} \Psi^{-1}(1-\alpha_k) \right\}\\
&= \bigvee_{k=1}^{\infty} \alpha_k.
\end{aligned}$$

Hence

$$\Upsilon(t) = \bigvee_{k=1}^{\infty} \alpha_k.$$

Substituting $t = T_F$, it can be obtained that

$$\text{Failure} = \Upsilon(T_F).$$

The theorem is proved.

5. Conclustion

This paper gave rigorous definitions of demand-driven uncertain production systems with delay effects and their performance margins. Furthermore, some formulas for computing the systems' belief reliability and first hitting time of failure were derived. The results provided practical mathematical tools for production planning, and gave valuable insights in reliability assessment as well as risk management for complicated systems considering delay effects.

Acknowledgement

This work was supported by the Funding of Science and Technology on Reliability and Environmental Engineering Laboratory, China (No.6142004220101).

References

Kang, R. (2020). *Belief reliability theory and methodology* (1st ed.). Beijing: National Defense Industry Press.

Lio, W. and L. Jia (2020). Uncertain production risk process with breakdowns and its shortage index and shortage time. *Journal of Intelligent & Fuzzy Systems 39*, 7151–7160.

Lio, W. and B. Liu (2020). Shortage index and shortage time of uncertain production risk process. *IEEE transactions on Fuzzy Systems 28*, 2856–2863.

Lio, W. and B. Liu (2021). Initial value estimation of uncertain differential equations and zero-day of covid-19 spread in china. *Fuzzy Optimization and Decision Making 20*, 177–188.

Liu, B. (2007). *Uncertainty Theory* (2nd ed.). Berlin: Springer-Verlag.

Liu, B. (2008). Fuzzy process, hybrid process and uncertain process. *Journal of Uncertain Systems 2*, 3–16.

Liu, B. (2009). *Some research problems in uncertainty theory*. Berlin: Springer-Verlag.

Liu, B. (2010). *Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty*. Berlin: Springer-Verlag.

Liu, B. (2013). Extreme value theorems of uncertain process with application to insurance risk model. *Soft Computing 17*, 549–556.

Liu, Y. and B. Liu (2021). Waiting time and idle time of uncertain queueing system. *International Journal of General Systems 50*, 871–890.

Liu, Y. and B. Liu (2022). Residual analysis and parameter estimation of uncertain differential equations. *Fuzzy Optimization and Decision Making 21*, 513–530.

Liu, Y. and Z. Qin (2024). Moment estimation of uncertain autoregressive model and its application in financial market. *Communications in Statistics - Simulation and Computation*, DOI: 10.1080/03610918.2024.2378113.

Liu, Y., Z. Qin, and X. Li (2024). Are the queueing systems in practice random or uncertain? evidence from online car-hailing data in beijing. *Fuzzy Optimization and Decision Making 23*, 497–511.

Liu, Z. and Y. Yang (2020). Uncertain insurance risk process with multiple classes of claims. *Applied Mathematical Modelling 83*, 660–673.

Wang, L. and X. Gao (2025). Least absolute deviations estimation for an uncertain moving average model. *Journal of Uncertain Systems 18*, Article 2450002.

Xie, J. and W. Lio (2024). Uncertain nonlinear time series analysis with applications to motion analysis and epidemic spreading. *Fuzzy Optimization and Decision Making 23*, 279–294.

Xie, J., W. Lio, and R. Kang (2024). Analysis of simple pendulum with uncertain differential equation. *Chaos, Solitons and Fractals 185*, Article 115145.

Yang, L. and Y. Liu (2024). Solution method and parameter estimation of uncertain partial differential equation with application to china's population. *Fuzzy Optimization and Decision Making 23*, 155–177.

Yang, X. and H. Ke (2023). Uncertain interest rate model for shanghai interbank offered rate and pricing of american swaption. *Fuzzy Optimization and Decision Making 22*, 447–462.

Yao, K. (2021). An uncertain single-server queueing model. *Applied Mathematical Modelling 14*, Article 2150001.

Yao, K. and X. Li (2012). Uncertain alternating renewal process and its application. *IEEE transactions on Fuzzy Systems 20*, 1154–1160.

Yao, K. and J. Zhou (2018). Ruin time of uncertain insurance risk process. *IEEE Transactions on Fuzzy Systems 26*, 19–28.

Yu, L., W. Lio, and R. Kang (2024). Uncertain delayed production risk process and its application in the shortage for a production process system. *International Journal of General Systems*, DOI: 10.1080/03081079.2024.2358347.

Zhang, X., Y. Ning, and G. Meng (2013). Delayed renewal process with uncertain interarrival times. *Fuzzy Optimization and Decision Making 12*, 79–87.