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Homotopy-series solution of eigenpair of random structure

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A new computational method is proposed for solving the eigenpair of the structure with random parameters based on the basic idea of the homotopy analysis method. For this new method, the eigenvalues and eigenvectors of the random structure are expressed as the homotopy-series. Because each term in the homotopy expressions includes an approaching function with auxiliary parameter *h*, the convergence domain of the homotopy-series is greatly improved, which makes this new method available for the large fluctuation of random parameters and different from the traditional Taylor series. Actually, the Taylor series is just a special case of the homotopy series where the value of *h* equals to -1. In practice, a dimension-reduction strategy is applied to the series to reduce the computational effort: the single-variable and double-variable homotopy-series are recommended for calculation. Numerical example, a fixed rectangular plate with random elastic moduli, indicates that the new method provides excellent approximations of the eigenpairs of a closely spaced eigenvalues system.

Keywords: Random eigenvalue problem, homotopy analysis method, Taylor series, stochastic finite element method.

1 Introduction

Algebraic eigenvalue problems are a class of basic and significant problems in various fields, such as structural dynamics and structural stability. Currently, the computation of eigenvalues and eigenvectors is well comprehended for deterministic problems (Ang and Amin 1968, Liu, Belyschko and Mani 1986). In many practical cases, however, the physical properties of the structural systems are not deterministic. Therefore, it is extremely necessary to use random variables to more realistically describe the uncertain characteristics that exist in eigenvalue problems in engineering (Huang and Li 2007).

Due to the randomness of the input parameters, such as the modulus of elasticity, of a physical problem, the desired output or eigenvalues will also be random. The methods for computing these random outputs are generally composed of two categories. The first category includes simulation-based methods. Direct Monte-Carlo simulation (DMC) is the most important and fundamental simulation-based method (Székely and Schuëller 2001, Shinozuka and Astill 1972), but it requires considerable computational effort, especially for large systems. The second category for random analysis, stochastic finite element methods (SFEM) (Schuëller 1997), primarily involves expansion-based methods. In this category, the main focus is perturbation methods (Kleiber and Hien 1992, Kamiński 2015) and spectral methods (Ghanem and Spanos 1991). Collins and Thompson (1969) presented a first-order perturbation method for

dynamic analysis of structures with parameter uncertainties. The popularity of this method can be primarily attributed to its ease of implementation and computational efficiency. However, the low-order perturbation method only gives reasonable results for statistical moments when the coefficients of variation of the random system parameters are small. Second-order or even high-order perturbation methods are instead chosen, and they improve this situation to allow a wider fluctuation range of random parameter (Kamiński 2015). In the case of the spectral method, the random eigenvalues and eigenvectors are approximated by projecting them on an orthogonal polynomial basis. For instance, the method proposed by Ghosh and Ghanem (2004) rewrites the eigenvalue problem as a set of non-linear equations, which is solved using the Newton–Raphson algorithm. The advantage of the spectral method over the perturbation method is that the accuracy using a given order of basis function is considerably better. Although the computational effort of the spectral method is more expensive than that of the perturbation method, it is, in general, considerably less expensive than the simulation-based methods. Moreover, one method that hybridizes perturbation and PC expansion approaches is proposed and aims to improve the efficiency of PC algorithms for random eigenvalue problems by using results from the perturbation method and a size reduction of the solved equations (Pascual and Adhikari 2012).

Apart from the above two categories of stochastic methods, other methods involving random eigenvalue problems are provided in the literature. One of them is the dimensional decomposition method, which allows lower-variate approximations of eigenvalues and lower-dimensional numerical integration for statistical moments (Rahman 2006). Another one among them is stochastic reduced basis approximation. It suggests that formulations that use a global set of stochastic basis vectors to simultaneously approximate all of the desired eigenvalues and eigenvectors may lead to more accurate results (Nair and Keane 2003). In summary, the current studies attempt to find an approach that is sufficiently efficient and accurate to address the eigenvalue problems of large systems with large fluctuations of random parameters with Gaussian and/or Non-Gaussian distributions. Based on this starting point, a new approach is presented in this paper for solving the eigenvalue problem of the structure with random parameters on the basis of the homotopy analysis method (Liao and Sherif 2004). In this approach, an infinite multivariate series of the involved random variables is proposed to express the random eigenvalue or even the random eigenvector. The coefficients of the multivariate series are determined by means of the homotopy analysis method. The proposed method is independent of random parameters with small fluctuations through a suitable choice of the auxiliary parameter. However, in practice, the single-and double-variable approximations are employed to simplify the calculation. A numerical example indicates that by selecting an optimal auxiliary parameter, the suggested approximations can produce very accurate results of eigenpairs even for closely spaced eigenvalues system.

2 Homotopy approximate of a random eigenvalue

2.1 Homotopy construction of random eigenvalue equations

The eigenvalue problem of undamped or proportionally damped deterministic systems can be expressed by Eq. (1), which is referred to as the eigenvalue algebraic equation.

$$\mathbf{KU} - \lambda \mathbf{MU} = 0 \tag{1}$$

where λ and $\mathbf{U}$ are the eigenvalue and corresponding eigenvector of the dynamic system. $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and mass matrices, respectively.

If the random field of the modulus of elasticity is defined as the Karhunen-Loève expansion or composed of some independent random variables, the stiffness matrix of the structure with random parameters can be written as

$$\mathbf{K}(\xi) \doteq \mathbf{K}_0 + \sum_{i=1}^{n} \xi_i \mathbf{K}_i \tag{2}$$

where $\mathbf{K}_0$ is the deterministic matrix with respect to deterministic mean parameters. $\mathbf{K}_i$ is an $N \times N$-dimensional matrix. $\xi = \{\xi_1, \xi_2, \cdots \xi_n\}$ are the independent random variables. As a result, the eigenvalue and eigenvector are functions of these random variables.

Now, by using the basic conception of HAM, the zero-order deformation equation of the random eigenvalue, is constructed as

$$\begin{aligned} &(1-p)[\mathbf{K}_0 \mathbf{W}_{(2)}(\xi,h,p) - W_{(1)}(\xi,h,p)\mathbf{M}_0 \mathbf{W}_{(2)}(\xi,h,p) - \mathbf{K}_0 \mathbf{U}_0 + \lambda_0 \mathbf{M}_0 \mathbf{U}_0] \\ &= ph[(\mathbf{K}_0 + \sum_{i=1}^{n} \mathbf{K}_i \xi_i)\mathbf{W}_{(2)}(\xi,h,p) - W_{(1)}(\xi,h,p)\mathbf{M}_0 \mathbf{W}_{(2)}(\xi,h,p)] \end{aligned} \tag{3}$$

where $p \in [0,1]$ and $h \neq 0$. $\mathbf{K}_0$, $\mathbf{M}_0$, λ_0 and $\mathbf{U}_0$ are the mean of stiffness, mass, eigenvalues and eigenvectors, respectively. Let $W_{(1)}(\xi,h,p)$ and $\mathbf{W}_{(2)}(\xi,h,p)$ denote the homotopy constructions of the eigenvalue and the eigenvector, respectively, which are the functions of the random variables ξ_i, the auxiliary parameter *h* and the embedding parameter *p*.

Therefore, as the embedding parameter *p* increases from 0 to 1, $W_{(1)}(\xi,h,p)$ varies from the initial approximation λ_0 to the solution $\lambda(\xi,h)$ of the original Eq. (1), and correspondingly, $\mathbf{W}_{(2)}(\xi,h,p)$ changes from the initial approximation $\mathbf{U}_0$ to the solution $\mathbf{U}(\xi,h)$.

Take partial derivative of the zero-order deformation of Eq. (3) *m* times with respect to *p* so the *m*th-order deformation equations of Eq. (1) can be attained. In this way, by utilizing the Taylor theorem, the final expression of $W_{(1)}(\xi,h,p)$ can be an infinite multivariate Maclaurin series, as shown in Eq. (4).

$$\begin{aligned} W_{(1)}(\xi,h,p) &= W_{(1)}(\xi,h,0) + \sum_{m=1}^{\infty} \left(\frac{W_{(1)}^{[m]}(\xi,h,p)}{m!} \right) p^m \\ &= W_{(1)}(\xi,h,0) + \frac{W_{(1)}^{[1]}(\xi,h,p)}{1!} p + \frac{W_{(1)}^{[2]}(\xi,h,p)}{2!} p^2 + \frac{W_{(1)}^{[3]}(\xi,h,p)}{3!} p^3 + \frac{W_{(1)}^{[4]}(\xi,h,p)}{4!} p^4 + \ldots\ldots \end{aligned} \tag{4}$$

where $W_{(1)}^{[m]}(\xi,h,p)$ denotes the *m*th-order partial derivative of $W_{(1)}(\xi,h,p)$ with respect to *p*, while *p* equals zero. Then, letting $p=1$ will produce the infinite series solution of the eigenvalue $\lambda(\xi,h)$ in the original Eq. (1), and the series solution can be expressed as

$$\lambda(\xi,h)=W_{(1)}(\xi,h,1)=\lambda_0+\Phi_{m,1}(h)\sum_{i_1=1}^{n}\lambda_{i_1}\xi_{i_1}+\Phi_{m,2}(h)\sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\lambda_{i_1i_2}\xi_{i_1}\xi_{i_2}+$$
$$\Phi_{m,3}(h)\sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\lambda_{i_1i_2i_3}\xi_{i_1}\xi_{i_2}\xi_{i_3}+\cdots+\Phi_{m,k}(h)\sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\cdots\sum_{i_n=1}^{i_{n-1}}\lambda_{i_1i_2\cdots i_n}\xi_{i_1}\xi_{i_2}\cdots\xi_{i_n}+\cdots \quad (5)$$

where $\lambda_{\bullet}$ are the deterministic eigenvalue coefficients and $\Phi_{m,k}(h)$ (k=1,.., m) are presented in reference (Huang, Zhang and Phoon 2018).

2.2 Two approximations of the infinite multivariate series

To improve the calculation efficiency, two approximations of the infinite multivariate series are proposed as follows

$$\lambda(\xi,h)=\lambda_0^{(1)}+\Phi_{m,1}(h)\sum_{i=1}^{n}\lambda_i^{(1)}\xi_i+\Phi_{m,2}(h)\sum_{i=1}^{n}\lambda_{ii}^{(1)}\xi_i^2+\Phi_{m,3}(h)\sum_{i=1}^{n}\lambda_{iii}^{(1)}\xi_i^3+\cdots+\Phi_{m,k}(h)\sum_{i_1=1}^{n}\lambda_{ii\cdots i}^{(1)}\xi_i^n+\cdots \quad (6)$$

$$\lambda(\xi,h)=\lambda_0^{(2)}+\Phi_{m,1}(h)\sum_{i=1}^{n}\lambda_i^{(2)}\xi_i+\Phi_{m,2}(h)\sum_{i=1}^{n}\sum_{j=1}^{i}\lambda_{ij}^{(2)}\xi_i\xi_j+\Phi_{m,3}(h)\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\lambda_{ijk}^{(2)}\Gamma_{ijk}\xi_i\xi_j\xi_k+\cdots \quad (7)$$

which are named as HSFEM-1 and HSFEM-2, respectively.

3 Example

This example involves the calculation of closely spaced eigenvalues of a fixed rectangular plate, as shown in Figure 1. For the plate, the width l_1=3 m, the length l_2=3.01 m, the thickness t=0.1 m and the Poisson's ratio v=0.3. The plate is divided into three parts, in accordance with the difference of the elastic modulus. Figure 1 shows that there are three elastic moduli, $e1$, $e2$ and $e3$. It is assumed that the three values of elastic moduli are independent random variables with Beta distributions. Their means are 10×10^5 kN/m^4, 4×10^5 kN/m^4 and 5×10^5 kN/m^4, respectively, and their coefficients of variation are 0.25, 0.25 and 0.3, respectively. The finite element mesh of the plate contains 144 4-noded thin plate elements and 169 nodes. Each node has 3 DOF, including one deflection and two rotations. The first three eigenvalues of the mean system are 63.86 (rad/s)^2, 230.03 (rad/s)^2 and 231.81 (rad/s)^2, respectively. The second and third are closely spaced eigenvalues. The random eigenvalues and eigenvectors of the plate are calculated by HSFEM-1, HSFEM-2 and the direct Monte Carlo simulation. Considering that the eigenvectors are highly sensitive to the closely spaced eigenvalues, for the second eigenvalue and eigenvector, the zero-order coefficients of the two approximations are calculated by Haichang Hu's method (Hu 1987). Then, the higher order coefficients of the two approximations are determined by the proposed method. The direct Monte Carlo simulation used 10,000 samples. Figures 2 are the probability density functions of the second eigenvalues. It is observed from Figures 2 that compared with HSFEM-1, HSFEM-2 yields significantly improved results that are in close agreement with those generated by the direct Monte Carlo simulation.

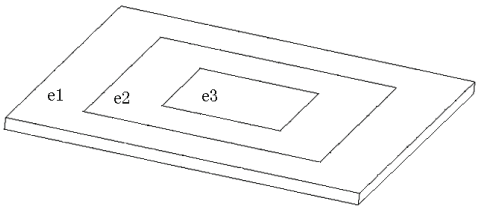


Figure 1. A fixed rectangular plate with random elastic moduli

Figures 3~4 show the means and standard variances of the modal shape of the second mode. The modal shape only considers the deflection of each node, and all the sample modal shapes calculated by the direct Monte Carlo simulation, are normalized. Figures 3 show that for the means of the modal shape of the second mode, the results of HSFEM-1, HSFEM-2 and the direct Monte Carlo simulation agree with each other very well. Alternately, Figures 4 indicates that compared with DMC, the standard variances of the modal shape of HSFEM-2 improve on those of HSFEM-1 and that the accuracy of their results is very good.

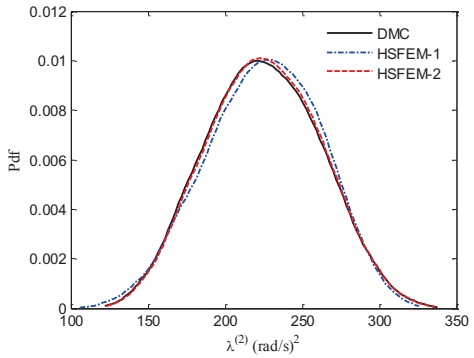
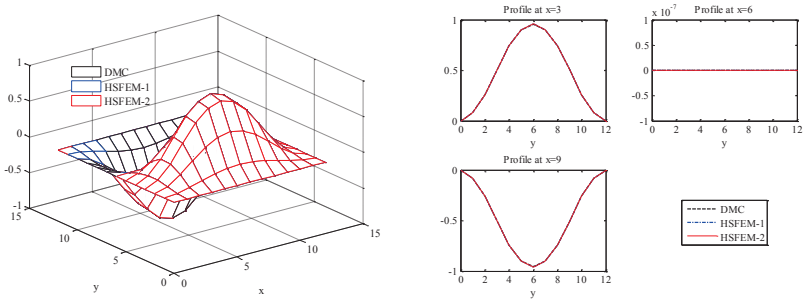
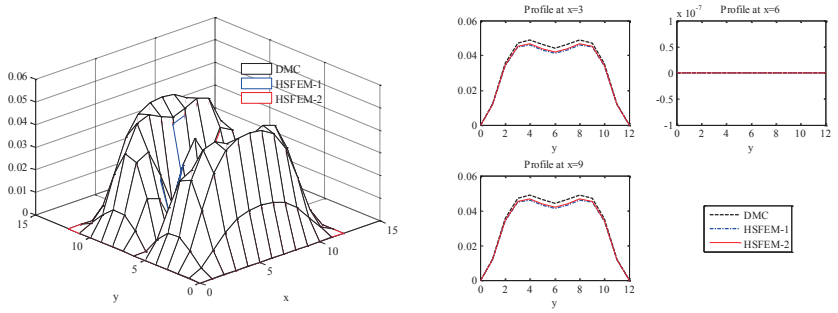


Figure 2. The PDFs of the second eigenvalue



(a) (b)

Figure 3. The mean of the modal shape of the second mode: (a) Curve of the mean modal shape; (b) Profiles at x=3, 6 and 9



(a) (b)

Figure 4. The standard variance of the modal shape of the second mode: (a) Curve of the mean modal shape; (b) Profiles at x=3, 6 and 9

4 Conclusion

A new approach, the homotopy stochastic finite element method, is established to compute the eigenvalues of a structure with random parameters on the basis of the homotopy analysis method. In this method, the random eigenvalues are expressed as an infinite multivariate series with respect to the involved random variables. Further, two approximations are proposed to simplify the calculation. Numerical studies indicated that the suggested approximations can produce very accurate results compared with the direct Monte Carlo simulation. In addition, the proposed methods are suitable for solving the closely spaced eigenvalue problem. Therefore, the proposed approach is a very good alternative method for solving random eigenvalue problems.

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