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STOCHASTIC RESPONSE ANALYSIS OF STRUCTURES USING ISOGEOMETRIC METHOD

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In this paper, based on stochastic perturbation technique, the deterministic isogeometric analysis (IGA) is extended to the stochastic framework, and the stochastic IGA method for random response analysis of structures is proposed. By representing random field with Karhunen- Loève expansion, stochastic IGA formulas are established. The first two moments of structural responses are formulated with stochastic perturbation method in the framework of IGA. Then, two numerical examples are demonstrated. The mean values and standard deviations of cantilever beam solved by using the stochastic IGA method are scrutinized by those by semi-analytical method and Monte Carlo simulation (MCS). The second example, Mindlin plate, is considered under different loading cases, whose results by using the stochastic IGA method are verified by stochastic finite element method (SFEM) and MCS. All the results of stochastic IGA in two numerical examples are in great agreement with those of MCS, indicating that stochastic IGA based on perturbation theory can achieve accurately and efficiently the stochastic responses of structures. In addition, from the results of Mindlin plate it is seen that the coefficient of variation of the response is insensitive to the loading conditions.

Keywords: Stochastic structural analysis, perturbation method, stochastic isogeometric analysis, random field, response moments.

1 Introduction

As is well known, there exist the various uncertainties of loads, geometric and material properties in realistic engineering systems. However, in traditional deterministic analysis, the uncertainty quantification and propagation of structures are not addressed. Therefore, the structures designed by traditional deterministic approaches may encounter the failure risk under uncertainty situation. To develop the computational methods for structural analysis and design considering random uncertainties, namely, the procedures of computational stochastic mechanics, becomes an important research topic (Oden et al. 2003). With extending the deterministic finite element method (FEM) to the stochastic framework, stochastic finite element method (SFEM) is proposed as a powerful tool to solve stochastic analysis problems of structures. Currently, SFEM has been extensively applied in a great variety of problems including solid, structural and fluid mechanics, acoustics and heat transfer (Stefanou 2009). Both static (Xia et al. 2014) and dynamic (Soize 2013) cases can be solved by this method.

Classical FEM usually adopts the polynomial interpolation functions, representing the physical field, to simulate the geometric shape, which, undoubtedly, leads to the inaccurate representation of curved geometric shape. Besides, it is not easy to obtain high-order continuity between elements in FEM, which adds the difficulties in constructing the plate and shell elements. To describe geometric and analytical models uniformly, and to seamlessly integrate CAD (Computer Aided Design) and CAE (Computer Aided Engineering), Hughes et al. (2005) proposed a novel numerical method, isogeometric analysis (IGA) based on spline theory. With some inherent advantages, IGA is considered as an alternative promising method for structural analysis. In the framework of IGA, the spline basis functions, describing the geometry exactly, are used as shape functions to represent the physical field simultaneously. Therefore, the geometry for structural analysis is exact. And the high-order continuity between elements is easy to be guaranteed by *k*-refinement. In addition, since there is no need to exchange data with the CAD system in the process of mesh refinement, adaptive analysis becomes easy to implement. IGA has been developed rapidly and employed successfully to address various physical and mathematical problems, for example, continuum mechanics, fluid dynamics, diffusion, and so on (Cottrell et al. 2009).

Based on the theoretical and application significance of uncertainty quantification, and the advantages of IGA over FEM, this work attempts to extend the deterministic IGA to the stochastic framework, and propose the numerical method of stochastic IGA to contribute for the solution of stochastic problems in structural analysis.

2 Representation of Random Field

Considering the research scope of this paper, Gaussian assumption for random field is made in the following. Karhunen-Loève (K-L) expansion (Ghanem and Spanos 1991) used in the majority of literature is also adopted. By K-L expansion, the Gaussian random field $H(\mathbf{x},\ \theta)$ can be represented in the following form

$$H(\mathbf{x},\theta) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \sigma(\mathbf{x})\sqrt{\lambda_i}\,\xi_i(\theta)\varphi_i(\mathbf{x}) \tag{1}$$

where $\mathbf{x}$ stands for the points in the random field domain and θ indicates the underlying random quantities; $\mu(\mathbf{x})$ and $\sigma(\mathbf{x})$ denote the mean and standard deviation of the random field, respectively; $\xi_i(\theta)$, $(i = 1,\ 2,\ ...,n)$ is a set of independent standard normal variables; λ_i and $\varphi_i(\mathbf{x})$ signify the *i*th eigenvalue and eigenfunction of the autocovariance function, respectively, which can be obtained by solving the Fredholm integral equation of the second kind

$$\int_{\Omega} \mathrm{Cov}(\mathbf{x},\mathbf{x}')\varphi_i(\mathbf{x}')\mathrm{d}\mathbf{x}' = \lambda_i\varphi_i(\mathbf{x}) \tag{2}$$

where $\mathrm{Cov}(\mathbf{x},\mathbf{x}')$ is the autocovariance function of points $\mathbf{x}$ and $\mathbf{x}'$ in the random field domain Ω. The autocovariance function, also referred to as kernel function, can be expressed in the form of $\mathrm{Cov}(\mathbf{x},\mathbf{x}') = \sigma(\mathbf{x})\sigma(\mathbf{x}')\rho(\mathbf{x},\mathbf{x}')$, in which $\rho(\mathbf{x},\mathbf{x}')$ is known as the autocorrelation coefficient function.

3 Isogeometric Analysis for Structure

3.1 *NURBS Basis Functions, NURBS Curves and Surfaces*

Underlying B-splines curves and surfaces, B-splines basis functions, for one dimension, are defined on knot vectors consist of a sequence of non-decreasing parameterized coordinates. For a knot vector $\Xi = \{\xi_1, \xi_2, ..., \xi_{n+p+1}\}$, $\xi_i \in \mathrm{R}$, $i = 1,2,...,n+p+1$, is called the *i*th knot, and

$[\xi_i, \xi_{i+1}]$ is termed as a knot span. Also, p denotes the order of polynomial basis functions, while n signifies the number of basis functions constituting relevant B-splines curves.

Given a knot vector, the B-splines basis functions can be obtained by the Cox-de Boor recursion formula

$$N_{i,0}(\xi) = \begin{cases} 1, \text{ if } \xi_i \le \xi \le \xi_{i+1} \\ 0, \text{ otherwise} \end{cases} \tag{3}$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \qquad p \ge 1 \tag{4}$$

with i =1, …, $n + p + 1$. It should be pointed out that the basis functions are $p - r_i$ times continuous differentiable at a knot ξ_i with r_i multiplicity.

By adding different weights w_i to every B-splines basis function $N_{i,p}(\xi)$, NURBS (nonuniform rational B-spline) basis functions are defined as

$$R_i^p = \frac{N_{i,p}(\xi) w_i}{\sum_{i'=1}^{n} N_{i',p}(\xi) w_{i'}} \tag{5}$$

Then, a NURBS curve is described as

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} R_i^p \mathbf{B}_i \tag{6}$$

where $\mathbf{B}_i \in \mathrm{R}^d (i = 1, 2, ..., n)$ represent the control points related to the basis functions R_i^p. Through the tensor product of two coordinate directions ξ and η, NURBS surfaces can be obtained.

3.2 *Isogeometric Analysis Based on NURBS*

In the framework of IGA, the NURBS basis functions describing the geometry exactly are used as shape functions to represent the physical field simultaneously. Namely, the geometric shape and the discretization of the field value adopt the same NURBS basis functions

$$\mathbf{x} = \sum_{I=1}^{n} R_I(\xi) \mathbf{B}_I \quad , \qquad \mathbf{u}(\mathbf{x}) = \sum_{I=1}^{n} R_I(\xi) \mathbf{u}_I \tag{7}$$

where $R_I(\xi)$ is NURBS basis function, and $\mathbf{u}_I$ denotes the vector of unknown displacements of the control point $\mathbf{B}_I$.

4 Stochastic Isogeometric Analysis

4.1 *Stochastic Equilibrium Equation*

For linear elastic continuum, the static stochastic equilibrium equation can be written as

$$\mathbf{K}(\theta)\mathbf{U}(\theta) = \mathbf{F} \tag{8}$$

where $\mathbf{K}(\theta)$, $\mathbf{U}(\theta)$, and $\mathbf{F}$ indicate the global stiffness matrix, the global displacement vector and the load vector, respectively. $\mathbf{K}(\theta)$ is obtained by assembling the element stiffness matrices $\mathbf{K}=\sum_{e\in\Omega}\mathbf{K}_e$. And the element stiffness matrix is calculated through

$$\mathbf{K}_e=\int_{\Omega_e}\mathbf{B}^{\mathrm{T}}\mathbf{D}(\theta)\mathbf{B}\mathrm{d}\Omega_e=\int_{\Omega_e}H(\mathbf{x},\theta)\mathbf{B}^{\mathrm{T}}\mathbf{D}_0\mathbf{B}\mathrm{d}\Omega_e \tag{9}$$

where $\mathbf{B}$ means the strain-displacement matrix, and $\mathbf{D}(\theta)$ indicates the random elastic matrix. However, $\mathbf{D}_0$ is the elastic constant matrix computed by setting Young's modulus to be 1 unit.

4.2 *Perturbation Based Stochastic Isogeometric Analysis*

In this work, the stochastic perturbation method (Kleiber and Hien 1992), usually based on the Taylor series expansion, is employed to calculate the statistical moments of the structural responses. The first order Taylor series expansions of $\mathbf{K}(\theta)$ and $\mathbf{U}(\theta)$ with respect to their means are given as

$$\mathbf{K}(\theta)\approx\bar{\mathbf{K}}+\sum_{i=1}^{M}\left.\frac{\partial\mathbf{K}(\theta)}{\partial\xi_i}\right|_{\boldsymbol{\xi}=\mathbf{0}}\xi_i \quad , \qquad \mathbf{U}(\theta)\approx\bar{\mathbf{U}}+\sum_{i=1}^{M}\left.\frac{\partial\mathbf{U}(\theta)}{\partial\xi_i}\right|_{\boldsymbol{\xi}=\mathbf{0}}\xi_i \tag{10}$$

where $\bar{\mathbf{K}}$ and $\bar{\mathbf{U}}$ denote the mean values of the stiffness matrix and displacement vector, respectively.

Substituting Eq.(10) into stochastic equilibrium equation (8), and comparing the coefficients of ξ_i on both sides, the following relations can be easily obtained

$$\bar{\mathbf{U}}=\bar{\mathbf{K}}^{-1}\mathbf{F} \quad , \qquad \frac{\partial\mathbf{U}(\theta)}{\partial\xi_i}=-\bar{\mathbf{K}}^{-1}\left.\frac{\partial\mathbf{K}(\theta)}{\partial\xi_i}\right|_{\boldsymbol{\xi}=\mathbf{0}}\bar{\mathbf{U}} \tag{11}$$

Finally,

$$E\,[\mathbf{U}(\theta)]=\bar{\mathbf{U}} \quad , \qquad \mathrm{Cov}[\mathbf{U}(\theta),\mathbf{U}(\theta)]=\sum_{i=1}^{M}\left.\frac{\partial\mathbf{U}(\theta)}{\partial\xi_i}\right|_{\boldsymbol{\xi}=\mathbf{0}}\left.\frac{\partial\mathbf{U}^{\mathrm{T}}(\theta)}{\partial\xi_i}\right|_{\boldsymbol{\xi}=\mathbf{0}} \tag{12}$$

are acquired as the mean vector and covariance matrix of the displacement vector $\mathbf{U}(\theta)$.

5 Numerical Examples of Stochastic IGA

5.1 *Cantilever Beam*

As shown in Fig.1, a two-dimensional cantilever beam with dimensions L = 10 units and D = 2 units is considered (Long et al. 2016). The left edge is fixed and the right edge is subjected to a shear load p = 150 units. The Young's modulus E of the beam is modeled as Gaussian random field with the mean μ = 3e7 units and the standard variance σ = 3e6 units. Poisson's ratio υ is equal to 0.25. The plane stress condition is assumed. The autocovariance function is assumed to be the exponential type. The correlation length along the x and y directions are set to be 7 units and 1 unit, respectively.

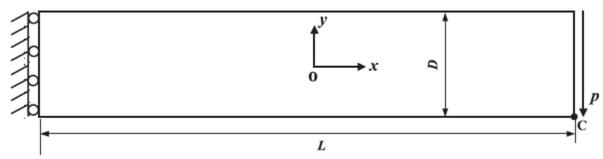

Figure 1. Cantilever beam

The first two orders of statistical moments, i.e., mean value and standard deviation, of the displacements at node **C** are calculated utilizing the semi-analytical method and stochastic IGA method proposed herein. Compared to MCS, as the semi-analytical method deals with structural random field analytically, it is believed that its solution is more accurate. The associated results are listed in Table 1, and it is observed that the stochastic IGA method has close agreement with the semi-analytical method, with the maximum error 0.05%, demonstrating that the stochastic IGA method is of high accuracy to deal with stochastic structural analysis. Also, the errors between stochastic IGA method and MCS are no more than 1.96%, which verifies the effectiveness of stochastic IGA method.

Table 1. The first two orders of statistical moments of the displacements at node **C**

	Semi-analytical solution	SIGA	Error (SIGA relative to semi-analytical)	MCS (Long et al. 2016)	Error (SIGA relative to MCS)
μ_x	-7.4554e-4	-7.4554e-4	0	-7.4787e-4	-0.31%
μ_y	-5.0533e-3	-5.0533e-3	0	-5.1054e-3	-1.02%
σ_x	4.7715e-5	4.7692e-5	-0.05%	4.8224e-5	-1.10%
σ_y	3.1292e-4	3.1302e-4	0.03%	3.1927e-4	-1.96%

5.2 *Mindlin Plate*

A Mindlin plate simply-supported on four edges, with dimensions 10 × 10 units, is considered here. Its thickness is 0.1 unit and Poisson ratio equals 0.3. The elastic modulus is modeled as Gaussian random field with the mean μ = 1.092e6 units and the coefficient of variation V= 0.1. The correlation length along x is 5 units as same as along y.

Table 2. The coefficient of variation of the response under uniformly distributed force

Coefficient of variation	SFEM	SIGA	MCS
W_A	0.0606	0.0611	0.0614
Θ_{xB}	-0.0618	-0.0611	-0.0628

Table 3. The coefficient of variation of the response under a concentrated force

Coefficient of variation	SFEM	SIGA	MCS
W_A	0.0641	0.0639	0.0646
Θ_{xB}	-0.0615	-0.0609	-0.0619

The coefficients of variation of the deflection at the central point **A** (x=5, y=5), and of the rotation at one boundary midpoint **B** (x=0, y=5), are selected to calculate under different loading conditions. In the first case, the uniformly distributed force with magnitude of 10 is acted on the whole plate. The coefficients of variation of the responses in two degrees of freedom are listed in Table 2. When only a concentrated force of 10 units is applied at the central point, the corresponding results are given in Table 3. Under both cases, the results of stochastic IGA are all in great agreement with those of SFEM and MCS. Meanwhile, they are smaller than the coefficient of variation of elastic modulus, which is equal to 0.1. Comparing Table 2 with Table 3, it is reasonable to argue that the loading conditions do not yield remarkable effect on coefficient of variation of the response.

6 Conclusions

This work proposes the stochastic isogeometric analysis method for random response analysis of structures, by extending the deterministic IGA to stochastic framework. Moreover, the stochastic IGA formulas for the first two moments of structural responses are established. Two numerical examples demonstrated that the results of stochastic IGA are in great agreement with those of MCS, and stochastic IGA based on perturbation theory can achieve accurately and efficiently the stochastic responses of structures. In addition, the coefficient of variation of the responses are smaller than that of elastic modulus. According to the results of Mindlin plate under different loading cases, it is reasonable to argue that the coefficient of variation of the responses is insensitive to loading conditions. Due to the theoretical and applicational significance of uncertainty quantification and the advantage of IGA over FEM, the stochastic IGA is an alternative promising method for addressing the stochastic analysis problems of engineering.

Acknowledgments

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