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STOCHASTIC SEISMIC ANALYSIS BY BIVARIATE GAUSSIAN MIXTURE BASED EQUIVALENT LINEARIZATION METHOD

SANG-RI YI[1], ZIQI WANG[2] and JUNHO SONG[3]

[1]*Dept. of Civil & Environmental Engineering, Seoul National University, Seoul, Korea.*
E-mail: yisangri@snu.ac.kr
[2]*Earthquake Engineering Research and Test Center, Guangzhou University, Guangzhou, China.*
E-mail: ziqidwang@yahoo.com
[3]*Dept. of Civil & Environmental Engineering, Seoul National University, Seoul, Korea.*
E-mail: junhosong@snu.ac.kr

To address technical challenges in stochastic seismic analysis of nonlinear systems, a Gaussian mixture based equivalent linearization method (GM-ELM) was recently developed. Unlike conventional equivalent linearization methods which identify a single equivalent linear system that meets certain equivalence response criteria, GM-ELM aims to capture the response probability distribution by stochastically decomposing a nonlinear system into multiple equivalent linear systems. Various response statistics of the nonlinear system, e.g. mean crossing rate, first passage probability and mean peak response, are estimated by recombining the response statistics of the equivalent linear systems computed based on linear random vibration theories. Retaining the original ideas of stochastic decomposition in GM-ELM, this paper proposes to use bivariate Gaussian mixture models instead of univariate ones. This is to improve accuracy by introducing additional dimension i.e. derivative of response, to the response probability density domain. Based on the use of bivariate model, the system decomposition and response combination of GM-ELM are reformulated. Moreover, to promote applications to performance based earthquake engineering (PBEE) practices, GM-ELM is extended for efficient fragility analysis by introducing auxiliary dimension related to the excitation intensity and multivariate Gaussian mixture model. The proposed methods are demonstrated by numerical examples.

Keywords: Random vibrations, nonlinear analysis, response spectrum, equivalent linearization.

1 Introduction

In efforts to evaluate seismic demand of structure, it is important to incorporate the uncertainty of ground motions into the analysis. To this end, many stochastic seismic analysis methods have been proposed, such as response spectrum or power spectrum based methods. However, these methods are not directly applicable to nonlinear systems, for which the superposition rule does not hold. To overcome challenges in such a *nonlinear* seismic analysis, various stochastic seismic analysis methods have been proposed, such as conventional equivalent linearization method (ELM), which finds an equivalent linear system that minimizes mean square error of response (Clough and Penzien 1975), and tail equivalent linearization method (TELM), which obtains equivalent linear system in terms of numerical impulse-response function by identifying "design point" from the first-order reliability method (FORM) (Fujimura and Der Kiureghian 2007).

More recently, a Gaussian mixture based equivalent linearization method (GM-ELM) was developed based on the approximation of the non-Gaussian distribution of the system response of interest by a group of Gaussian distributions (Wang and Song 2017). The basic presumption behind GM-ELM is that each Gaussian component in the identified mixture model can be interpreted as an imaginary linear single-degree-of-freedom (SDOF) oscillator. The point is that the response of original system is restored in terms of these equivalent linear systems, to which standard linear seismic analysis methods can be applied.

However, the original GM-ELM (Wang and Song 2017) may suffer from the fact that the number of unknown parameters is larger than the given pieces of information, and therefore, heuristic assumptions regarding the equivalent linear system are needed. In this research, we propose to import additional information from the probability density of time derivative of response to avoid the heuristic assumption on the structural damping. Also, the combination rules of equivalent linear systems are reformulated accordingly, which improves the accuracy. Another development introduced in this paper is a new efficient fragility analysis approach based on GM-ELM. This approach can identify global equivalent linear systems while not being influenced by the scale of the excitation.

The next section reviews the concepts of stochastic decomposition and extends the univariate GM-ELM into the bivariate version. Section 3 re-establishes mathematical equations to combine response statistics of equivalent linear systems for the bivariate GM-ELM, and Section 4 introduces a novel approach for efficient fragility analysis using GM-ELM. Numerical examples are provided in Section 5, followed by conclusions in Section 6.

2 Gaussian Mixture based Equivalent Linearization method (GM-ELM)

2.1 *Basic concepts of GM-ELM*

The GM-ELM identifies multiple equivalent linear systems that can capture the shape of the response probability density function (PDF) of a generic nonlinear system. From the practical and theoretical rationales that (a) random seismic excitation is often featured as a filtered Gaussian process (Fan and Ahmadi 1990, Fujimura and Der Kiureghian 2007), and (b) the corresponding response of a linear system is also a Gaussian process, GM-ELM identifies a group of imaginary SDOF oscillators which are randomly activated, constantly switching from one to another. This is done by introducing a Gaussian mixture model to approximate generic PDF of response of interest while each Gaussian model in the mixture is used to identify equivalent SDOF oscillators.

This process could be interpreted as stochastic decomposition or stochastic linearization. By evaluating response statistics of equivalent linear systems individually and then combining them with probabilistically derived combination rules, the response statistics of nonlinear system could be acquired within acceptable amount of computing power. Figure 1 provides the conceptual diagram on the analysis procedure by GM-ELM.

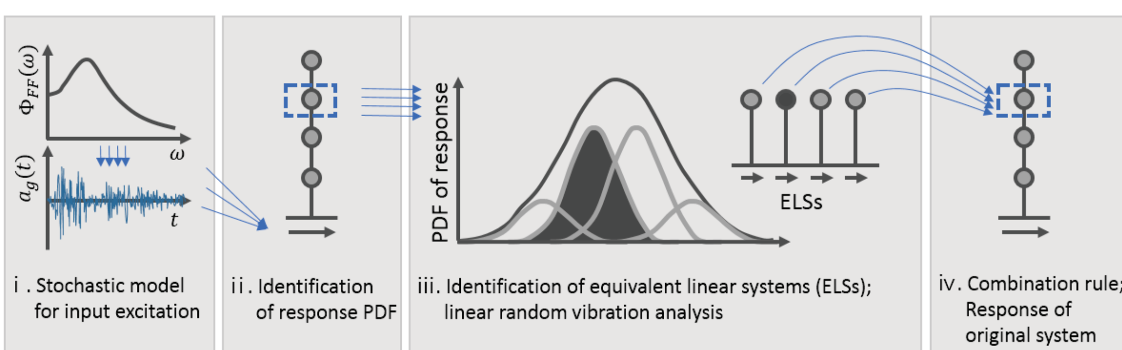

Figure 1. Conceptual illustration of analysis procedure by GM-ELM

2.2 *Details of stochastic decomposition*

To illustrate the proposed concept of stochastic decomposition, a stationary and ergodic filtered Gaussian process excitation is considered. The resulting stationary response could be represented in terms of PDF. As an extension of the original GM-ELM, we propose to consider the time derivative of the response as well as the response of interest, i.e., $\boldsymbol{z} = \{z, \dot{z}\}$.

$$p_{GM}(\boldsymbol{z}; \boldsymbol{v}) = \sum_{k=1}^{K} \alpha_k f_N(\boldsymbol{z}; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \tag{1}$$

where the function $f_N(\boldsymbol{z}; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$ represents uncorrelated bivariate Gaussian distribution with mean $\boldsymbol{\mu_k} = \{\mu_{z,k}, \mu_{\dot{z},k}\}$ and diagonal covariance matrix $\boldsymbol{\Sigma}_k$ that have diagonal terms $\{\sigma_{z,k}^2, \sigma_{\dot{z},k}^2\}$, and α_k is the mixture coefficient that describes the relative rate of occurrence of each mixture component. The parameters $\mathbf{v} = \{\alpha_k, \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k\}$ could be estimated by minimizing the cross-entropy of two distributions or by maximizing the likelihood of samples.

Each Gaussian mixture component represents the response of a linear system subjected to the Gaussian excitation. The means $\mu_{z,k}$ and $\mu_{\dot{z},k}$ represent the shifted amount of the base location and the base velocity of the corresponding imaginary single-degree-of-freedom (SDOF) oscillator, respectively while the variances $\sigma_{z,k}^2$ and $\sigma_{\dot{z},k}^2$ represent their stochastic response characteristics. From random vibration theories, the variances of response and its time derivative are related to the linear system parameters as follows:

$$\sigma_{z,k}^2 = \int_{-\infty}^{\infty} |H_k(\omega)|^2 S_f(\omega)\, d\omega, \qquad k = 1, \dots K \tag{2a}$$

$$\sigma_{\dot{z},k}^2 = \int_{-\infty}^{\infty} \omega^2 |H_k(\omega)|^2 S_f(\omega)\, d\omega, \qquad k = 1, \dots K \tag{2b}$$

where $S_f(\omega)$ is the auto power spectral density (PSD) of the seismic excitation, and $H_k(\omega)$ is the linear frequency response function (FRF) of the k-th component, i.e.

$$H_k(\omega) = \frac{s_{eq,k}}{k_{eq,k} + i\omega c_{eq,k} - m_{eq,k}\omega^2} \tag{3}$$

in which $k_{eq,k}$, $c_{eq,k}$, $m_{eq,k}$, and $s_{eq,k}$ are respectively the equivalent stiffness, damping, mass, and scaling factor. The scaling factor is defined according to the response quantity of interest. Therefore, to specify the equivalent linear systems, it is required to handle at least three unknown parameters while only two constraints, Eq. (2a) and Eq. (2b), are given. To address this, $m_{eq,k}$ can be set as the value of lumped mass at a location of interest. Note that the original approach of univariate GM-ELM only utilizes Eq. (2a), therefore one of the parameters, stiffness or damping, has to be pre-defined based on engineering judgements additionally.

3 Random Vibration Analysis by Bivariate GM-ELM

Various response statistics subject to seismic vibration can be calculated from the responses of identified equivalent linear systems. First, the instantaneous failure probability of a structure is defined as complementary cumulative distribution function (CCDF),

$$\Pr(Z \geq z) = \sum_{k=1}^{K} \Pr(I_k = 1) \Pr(Z \geq z | I_k = 1) = \sum_{k=1}^{K} \alpha_k \Phi\left(-\frac{z - \mu_{z,k}}{\sigma_{z,k}}\right) \tag{4}$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of standard normal distribution, and $I_k = 1$ indicates that the k-th equivalent SDOF is activated while others are not. This could be extended to the threshold of velocity as well when bivariate GM-ELM is used. For the mean up-crossing rate $\nu^+(z)$, the following expressions can be derived from theories of random vibration analysis (Lutes and Sarkani 2004) and an assumption about the "mode switch" (Wang and Song 2017):

$$\nu^+(z) = \sum_{k=1}^{K} \alpha_k \nu_k^+(z) \tag{5}$$

$$\nu_k^+(z) = \frac{1}{2\pi} \sqrt{\frac{\tilde{\lambda}_{2,k}}{\lambda_{0,k}}} \exp\left(-\frac{0.5\left(z - \mu_{z,k}\right)^2}{\lambda_{0,k}}\right) \tag{6}$$

where the j-th order spectral moment of k-th equivalent SDOF $\lambda_{j,k}$ and $\tilde{\lambda}_{2,k}$ can be respectively computed as

$$\lambda_{j,k} = \int_{\infty}^{\infty} |\omega|^j \, |H_k(\omega)|^2 S_f(\omega) d\omega, \quad j = 0,1,2 \tag{7}$$

$$\tilde{\lambda}_{2,k} = 2\pi \left(\varphi\left(\delta_{\dot{z},k}\right) + \delta_{\dot{z},k} - \delta_{\dot{z},k}\Phi\left(\delta_{\dot{z},k}\right)\right)^2 \lambda_{2,k} \tag{8}$$

where $\varphi(\cdot)$ and $\Phi(\cdot)$ are the PDF and CDF of the standard normal distribution, respectively, and $\delta_{\dot{z},k} = \mu_{\dot{z},k}/\sigma_{\dot{z},k}$. Note that when $\mu_{\dot{z},k}$ is zero for all components, Eq. (6) gives the same expression to the formulation of univariate GM-ELM proposed in Wang and Song (2017). Under an assumption that the up-crossing events follow a Poisson process with the mean rate $\nu^+(z)$, the first passage probability for a time period T_d can be estimated as

$$\Pr[\max Z(t) > z\,]_{t \in T_d} = 1 - A \exp(-\nu^+(z) T_d) \tag{9}$$

in which A denotes the probability of the initial response being in the safe domain.

4 Finding Global Equivalent Linear Systems of GM-ELM for Fragility Analysis

This section extends the concept of GM-ELM to facilitate fragility analysis of nonlinear structures. The apparent weakness of the GM-ELM in fragility calculations is that as the intensity of ground motion of interest is changed to construct a fragility "curve" or "function," it is naturally required to re-evaluate the response PDF for each intensity level and also identify a new equivalent linear system. This makes it inefficient to construct fragility curves using a GM-ELM approach.

In order to overcome this issue, it is desirable to obtain universal equivalent linear systems which are independent of the intensity of the ground motions. For this, an axillary dimension which represents the excitation intensity is introduced in addition to the domain of the response(s). Considerer a variable I that represents the scale of seismic excitation. Drawing an analogy to the conventional GM-ELM approaches, one could approximate the distribution $p_{GM}(\mathbf{z}, I; v)$ by a set of uncorrelated multivariate Gaussian densities with the parameter set $\mathbf{v} = \{\alpha_k, \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k\}$, where in this case we have 3-dimensional parameters, i.e. mean vector $\boldsymbol{\mu}_k = \{\mu_{z,k}, \mu_{\dot{z},k}, \mu_{I,k}\}$ and diagonal covariance matrix $\boldsymbol{\Sigma}_k$ with the diagonal components $\{\sigma_{z,k}^2, \sigma_{\dot{z},k}^2, \sigma_{I,k}^2\}$. Exploiting this

Gaussian mixture model expression, the conditional probability density function given a specific excitation intensity can be derived as

$$p_{\mathrm{GM}}(\boldsymbol{z}|I) = \frac{p_{GM}(\boldsymbol{z}, I; \boldsymbol{v})}{p_{GM}(I; \boldsymbol{v})} = \sum_{k=1}^{K} \tilde{\alpha}_k(I) f_N(\boldsymbol{z}; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) \tag{10}$$

which gives a bivariate Gaussian mixture model with the mixture coefficients,

$$\tilde{\alpha}_k(I) = \frac{\alpha_k f_N\left(I; \mu_{I,k}, \sigma_{I,k}^2\right)}{\sum_{m=1}^{K} \alpha_m f_N\left(I; \mu_{I,m}, \sigma_{I,m}^2\right)} \tag{11}$$

Note that while the parameters used for identifying equivalent system i.e. $\left\{\mu_{z,k}, \mu_{\dot{z},k}, \sigma_{z,k}^2, \sigma_{\dot{z},k}^2\right\}$, stay constant, only the occurrence frequencies of the equivalent systems, i.e. $\tilde{\alpha}_k(I)$, vary depending on the ground motion intensities. Eq. (10) implies that once the global equivalent linear systems are identified, ordinary combination rules for GM-ELM approach, Eq. (4)-(9), could be directly applied to identify response statistics of various ground motion scales, where only α_k should be modified into $\tilde{\alpha}_k(I)$. This feature specially gives advantage when evaluating fragility curve, allowing us to avoided tedious repetition of finding equivalent linear systems for each of the selected ground motion intensities.

4 Numerical Examples

Consider a hysteretic oscillator under seismic loading described by the differential equation

$$m\ddot{X}(t) + c\dot{X}(t) + k_o[\alpha X(t) + (1-\alpha)Y(t)] = -m\ddot{U}_g(t) \tag{12}$$

where $X(\mathrm{t}), \dot{X}(t)$, and $\ddot{X}(t)$ denote the displacement, velocity and acceleration, respectively, of the oscillator. The initial natural period is set as 0.75 seconds and the damping ratio is 0.03. The parameter $\alpha = 0.1$ defines the characteristic of hysteresis. The term $Y(t)$ follows the Bouc-Wen hysteresis relation, i.e.

$$\dot{Y}(t) = -\gamma\left|\dot{X}(t)\right||Y(t)|^{\bar{n}-1}Y(t) - \eta|Y(t)|^{\bar{n}}\dot{\mathrm{X}}(t) + A\dot{X}(t) \tag{13}$$

where $\bar{n} = 3$, $A = 1$, and $\gamma = \eta = 1/2u_y^{\bar{n}}$, in which $u_y = 0.0552$ m is related to the yielding displacement. The ground motion $\ddot{U}_g(t)$ in Eq. (12) is modeled by a stationary auto-PSD described by modified Kanai-Tajimi model (Cloud and Penzien, 1975).

$$S_f(\omega) = S_o \frac{\omega_f^4 + 4\zeta_f^2\omega_f^2\omega^2}{\left(\omega_f^2 - \omega^2\right)^2 + 4\zeta_f^2\omega_f^2\omega^2} \frac{\omega^4}{(\omega_s^2 - \omega^2)^2 + 4\zeta_s^2\omega_s^2\omega^2} \tag{14}$$

where S_{o} is a scale factor, and $\omega_f = 15\ \mathrm{rad/s}$, $\zeta_f = 0.6$, $\omega_s = 1.5\ \mathrm{rad/s}$, and $\zeta_s = 0.6$ are the filter model parameters. The duration of excitation is set as 17 seconds. To prove the necessity of introducing bivariate Gaussian mixture model, both uni- and bivariate GM-ELM are used with scale factor S_o =0.015 $\mathrm{m^2/s^3}$.

Figure 2 show the mean up-crossing rate and first passage probability estimated by 100 rounds of dynamic simulations. It is seen that the bivariate GM-ELM decreases the error by taking into account the damping of the hysteretic behavior accurately. On the other hand, Figure 3 provides the fragility curves generated by the bivariate GM-ELM and global equivalent linear system approach described in the previous section. For convenience, $P(I)$ is set to be uniform distribution along the range of interest. A total of 200 excitation intensities are sampled to perform dynamic simulations for each sampled intensity and identify the corresponding response PDFs. Gaussian mixture model $p_{GM}(\boldsymbol{z}, I; \boldsymbol{v})$ is then fitted to the samples. The results in Figure 3 confirm that the fragility curves obtained from the GM-ELM gives similar results to those by Monte Carlo simulations.

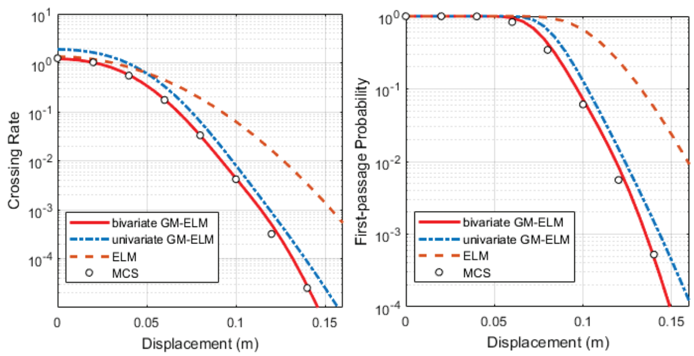

Figure 2 Mean up-crossing rates and first-passage probabilities

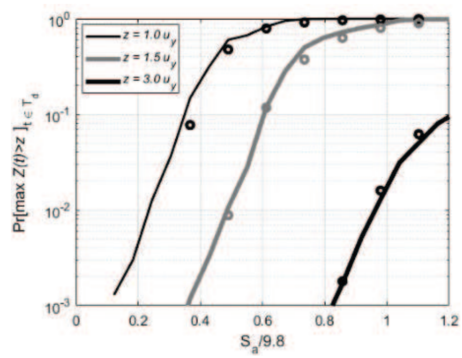

Figure 3. Fragility curves by GM-ELM; Markers represent MCS results based on 70,000 simulations

5 Conclusions

In this paper, a recently developed stochastic dynamic analysis approach, GM-ELM, is further developed. First, the response time derivative is additionally considered to identify the equivalent linear systems of GM-ELM more accurately. This is done by introducing a bivariate Gaussian mixture model instead of univariate one. Second, a concept of global equivalent linear system, which is invariant to the excitation intensity, is proposed by introducing auxiliary dimension related to the scale of ground motions and by approximating the extended PDF using multivariate Gaussian mixture model. It is shown that, by identifying the equivalent linear systems are invariant of the scale of ground motions only once and estimating their occurrence coefficients for each intensity level, one can still obtain accurate fragility curves. Further research is underway to develop GM-ELM-based methods for performance based earthquake engineering (PBEE).

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