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A MARGINAL UTILITY DAY-TO-DAY FLOW DYNAMICS FOR BOUNDEDLY RATIONAL USER EQUILIBRIUM

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This paper extends two existing day-to-day models, including the Smith dynamic (Smith, 1984) and the BNN dynamic (Brown and von Neumann, 1950), to capture the marginality and bounded rationality in traffic flow evolution. Properties of the extended Smith dynamic are theoretically investigated and the extended BNN dynamic is numerically evaluated. We found the stationary point of the proposed MU-BRUE Smith dynamic coincides with boundedly rational user equilibrium (BRUE) path flow pattern. It is proved that this dynamic system is a RBAP-BRUE processes (Ye and Yang, 2017) and admit the globally asymptotical stability. Numerical experiments are conducted to compare the evolution trajectories with and without considering the marginal cost and examine the stability of proposed models.

Keywords: Day-to-day, boundedly rational user equilibrium, marginal utility, mean dynamic.

1 Introduction

Bounded rationality was first proposed by Herbert A. Simon in the 1950s to model people's irrational decision-making behaviors resulted from the cognition limitation. It was first introduced by Mahmassani and Chang (1987), and later widely investigated in the transportation research. As an extension of classical user equilibrium (UE), boundedly rational user equilibrium (BRUE) characterizes a stable status that travelers can take any path whose journey time is within a so-called "indifference band" of the shortest path cost (Di et al., 2013). These models focus on the final state of distribution of Origin-Destination (OD) demand throughout the entire network. While in a real network, it is always observed that events like traffic incidents, demand variation, capacity modifications etc. might affect the traffic flow pattern from equilibrium to disequilibrium (Kumar and Peeta, 2015). For such cases, the day-to-day models are required to explain the mechanism of network flow evolution and the possibility of approaching the UE or BRUE state. The empirical flow data during the collapse and reopening of the I-35 W Bridge suggested that a network change could be irreversible when the initial equilibrium path flow pattern cannot be restored by revoking the change (Guo and Liu, 2011). To account for this irreversible change, they proposed a link-based BRUE day-to-day model, whose mathematical properties were further investigated by Di (2013). Recently, Ye and Yang (2017) proposed a general framework for the BRUE-based "rational behavior adjustment process" (BRUE-RBAP).

Most day-to-day models focus on the impact of historical path flow and cost on the number of drivers switching their path under the dis-equilibrium state. Mankiw (2012) proposed the principle that "rational individual think at marginal utility" which stated that a rational people evaluate the marginal cost when there are multiple choices. Inspired by this principle, Kumar and Peeta (2015) incorporated the sensitivity of the path cost to path flow into the day-to-day model.

In this paper, we extend two existing day-to-day models, including the Smith dynamic (1984) and the BNN dynamic (1950), to capture both the marginality and the bounded rationality in the flow evolution process towards equilibrium. The invariance property and asymptotic stability are discussed in this study. The rest of this paper is organized as follows. Section 2 introduces the background of mean-dynamics. Section 3 introduces the two existing models. Theoretical properties of the extended smith dynamic are analyzed. In Section 4, numerical example based on the Braess network is presented to compare the evolution processes of extended Smith dynamics and BNN dynamics with different initial states.

2 Mean-dynamics

We first provide the notations and definitions. The feasible flow set are defined as $\Omega \triangleq \{\mathbf{x}|\mathbf{x}= \mathbf{\Delta f}, \mathbf{d} = \mathbf{\Lambda f}, \mathbf{f} \geq \mathbf{0}\}$, where $\mathbf{\Delta}$ is the link-path matrices and $\mathbf{\Lambda}$ is the OD-path matrices.

Notation	Definition
W	the set of O-D pairs;
R^w	the set of all paths connecting OD pair $w \in W$;
d^w	travel demand for O-D pair $w \in W$;
f_p^w	path flow on path $p \in R^w$;
$\mathbf{f}$	the vectors of path flow $\mathbf{f} = \left(f_p^w, p \in R^w, w \in W\right)^{\mathrm{T}}$;
v_a	link flow on link $a \in A$, where A is the link set;
$\mathbf{v}$	the vectors of link flows $\mathbf{v} = (x_a, a \in A)^{\mathrm{T}}$;
c_p^w	path travel cost on path p for OD-pair w;
μ^w	the minimum path travel cost between OD pair $w \in W$.

Based on the notion of revision protocol, Sandholm (2010) developed the framework of mean dynamic Eq. (1). Many existing deterministic evolutionary game models can be viewed as its special cases. The similar framework in the perspective of day-to-day dynamics was introduced by Yang and Liu (2005).

$$\frac{df_p^w}{dt} = \sum_{q \in R_w} f_q^w \rho_{qp,w} - f_p^w \sum_{q \in R_w} \rho_{pq,w}, \forall q \in R_w, w \in W, \tag{1}$$

where $\rho_{qp,w}(\mathbf{c}_w, \mathbf{f}_w)$ describes the swapping rate of each driver from path q to path p connecting OD pair w. It has been shown that when the revision protocol follows $\rho_{qp,w} = k_w \max\left(0, c_q^w - c_p^w\right)$, where k_w is an OD-dependent sensitivity parameter, we obtain the Smith dynamic. If $\rho_{qp,w} := k_w \tau_{pw}$, where $\tau_{pw}(\mathbf{f}) = P\left(p = \operatorname*{argmin}_{q \in R_w} c_q^w(\mathbf{f})\right)$, the mean dynamic degenerate to the best response dynamics. Even though the best response dynamics is discontinuous, the existence result may still exist due to the good regularity properties of the set-valued mapping. If the revision protocol follows $\rho_{qp,w} := \alpha_w \tau_{pw}$, where $\tau_{pw}(\mathbf{f}) = \max\left\{0, -c_p^w + \bar{c}_w\right\}$, we arrive at the BNN dynamic. Eq. (1) describes that the current path cost c and flow f both influence the switching rate under the dis-equilibrium state. Moreover, in real-world scenario, drivers may also consider the sensitivity of path cost to the flow (He and Peeta, 2015). Drivers would more likely to shift to a path whose cost increases slowly with flow, e.g., the arterials with large capacity. Combining the above three factors, Kumar and Peeta (2015) furnished a MU-based dynamic model in which the path set connecting an OD pair is divided into sets of expensive path $P^w$and attractive path $\bar{P}^w$ using the shortes path travel time μ^w and a threshold δ. The path flow will outflow from the expensive set and be distributed to the paths in attractive set according to the marginal cost $mc_p^w = \frac{dc_p^w}{df_p^w}$.

$$\dot{f}^w(t)=\begin{cases}-\lambda^w f_k^w(c_k^w-\mu^w), & \forall k\in P^w,\forall w\in W\\ \dfrac{\sum_{k\in P^w} f_k^w(c_k^w-\mu^w)}{mc_l^w(f)\sum_{k\in\bar{P}^w}\dfrac{1}{mc_k^w}}, & \forall l\in\bar{P}^w,\forall w\in W\end{cases}. \quad (2)$$

However, the RHS of Eq. (2) may not satisfy the Lipstiz continuity and thus may have no classical solution, which is revealed by the following toy example. Consider a single network with only one OD pair connected by three parallel links. The OD demand is 12 and the link cost functions are given by $c_1=\frac{1}{2}f_1^2+2$, $c_2=f_2^2+13$, $c_3=5f_3+11$, respectively. Suppose the link flow pattern is $\mathbf{f}=(6,3,3)$, which is not an equilibrium state. If δ is slightly greater than $\frac{1}{3}$, then attractive set $\bar{P}=\{1,2\}$. By Eq. (2), $\dot{\mathbf{f}}=(9,9,-18)$ with $\lambda=1$. The flow of link 1 and 2 will be increased by same amount during a tiny time interval, after which the link 2 will immediately become unacceptable since the marginal cost of link 2 is greater than link 1. However, once link 2 joins the unacceptable set P, its flow will decrease immediately and rejoin the attractive set $\bar{P}$. So the RHS of Eq. (2) may not satisfy the Lipschitz continuity. When the cost function is affine, the discontinuity will not occur. This discontinuity also occurs in Di (2013), which is revealed by Ye and Yang (2017). To guarantee the nice property, the sensitivity parameter λ^w must be bounded above by the largest excess cost of a path (see Property 1 in Kumar and Peeta (2015)).

3 Marginal Utility Day-to-day Dynamics Formulation for BRUE

To avoid the disturbing discontinuity, we extend some well-defined existing dynamical models to capture both the marginality and the bounded rationality. By introducing the marginal term, we rewrite the Eq. (1) as:

$$\frac{df_p^w}{dt}=\sum_{q\in R_w}\frac{f_q^w}{mc_p^w}\rho_{qp,w}-\frac{f_p^w}{mc_q^w}\sum_{s\in R_w}\rho_{pq,w}, \forall q\in R_w, w\in W \quad (3)$$

This form has the following features: (1)The revision revision protocol $\rho(\mathbf{c}_w,\mathbf{f}_w)$ decides whether it is necessary to switch the routes. If the drivers have no alternative choice, they will not consider the marginal cost; (2)When the drivers have multiple alternatives, the marginal cost and the travel cost will combine to affect the switch rate. A path with higher marginal cost will be less attractive.

In general, we can extend the revision protocol $\rho(\mathbf{c}_w,\mathbf{f}_w)$ to $\rho(\mathbf{c}_w,\mathbf{f}_w,\boldsymbol{\epsilon}_w)$ by considering the indifference band $\boldsymbol{\epsilon}_w$ for BRUE. Suppose ρ follows the Lipschitz condition, Eq. (3) can admit a unique solution. To exemplify this extension, we will focus on the Smith dynamic, where revision protocol $\rho_{qp,w}=k_w\max(0,c_q^w-c_p^w-\varepsilon^w)$. Note that this extension can also be made for other dynamics following the mean dynamic. For example, the MU-BRUE based BNN dynamic can be achieved by revising $\rho_{qp,w}=\alpha_w\tau_{pw}$, where $\tau_{pw}(\mathbf{f})=\max\{0,-c_p^w+\bar{c}_w-\varepsilon^w\}$ and $\bar{c}_w$ is the average travel time of all the paths of OD pair w. MU-BRUE based smith dynamic can be described by:

$$\frac{df_p^w}{dt}=k_w\left(\sum_{q\in R^w}\left(\frac{f_q^w}{mc_p^w}\left(c_q^w-c_p^w-\varepsilon^w\right)_+-\frac{f_p^w}{mc_q^w}\left(c_p^w-c_q^w-\varepsilon^w\right)_+\right)\right), p\in R^w, w\in W. \quad (4)$$

Eq. (4) indicates that when drivers make decisions, path cost and the indifference band both determine whether it is a necessary swap between the two routes. If the drivers have no incentive to switch the route, marginal cost will not be considered. When the drivers have more than one option, the marginal cost will also influence the swapping rate. We first show Eq. (4) satisfies the important invariance property, which is expected for any valid fixed demand day to day models.

Invariance property. *If* $\mathbf{f}(0)\in\Omega$, *it holds that* $\mathbf{f}(t)\in\Omega,\forall t>0$.

Proof: From Eq. (3), we have:

$$
\begin{aligned}
\sum_{p\in R_w}\frac{df_p^w}{dt} &= \sum_{p\in R_w} k_w\left(\sum_{q\in R_w}\left(\frac{f_q^w}{mc_p^w}\left(c_q^w-c_p^w-\varepsilon^w\right)_+ - \frac{f_p^w}{mc_q^w}\left(c_p^w-c_q^w-\varepsilon^w\right)_+\right)\right)\\
&= k_w\left(\sum_{p\in R^w}\sum_{q\in R_w}\frac{f_q^w}{mc_p^w}\left(c_q^w-c_p^w-\varepsilon^w\right)_+ - \sum_{p\in R_w}\sum_{q\in R_w}\frac{f_p^w}{mc_q^w}\left(c_p^w-c_q^w-\varepsilon^w\right)_+\right).
\end{aligned} \tag{5}
$$

By rewriting the second term in Eq. (5), we have:

$$\sum_{p\in R_w}\sum_{q\in R_w}\frac{f_p^w}{mc_q^w}\left(c_p^w-c_q^w-\varepsilon^w\right)_+ = \sum_{q\in R_w}\sum_{p\in R_w}\frac{f_q^w}{mc_p^w}\left(c_q^w-c_p^w-\varepsilon^w\right)_+. \tag{6}$$

We can regard the $\frac{f_q^w}{mc_p^w}\left(c_q^w-c_p^w-\varepsilon^w\right)_+$ as an element in matrix where p and q denote the row and column respectively. Hence we have:

$$\sum_{p\in R_w}\sum_{q\in R_w}\frac{f_q^w}{mc_p^w}\left(c_q^w-c_p^w-\varepsilon^w\right)_+ - \sum_{q\in R_w}\sum_{p\in R_w}\frac{f_q^w}{mc_p^w}\left(c_q^w-c_p^w-\varepsilon^w\right)_+ = 0. \qquad \blacksquare$$

Next we show that the fixed point (set) of the dynamical system described by Eq. (4) coincide with the BRUE, which is elaborated in ***Theorem 1***. Following Guo and Liu (2011), the BRUE condition is given below.

Definition of BRUE. *A path flow vector* **f** *is said to be a boundedly rational user equilibrium (BRUE) path flow pattern Ω^* if the following condition holds:*

$$c_p^w(\mathbf{f})\begin{cases}\le \mu^w+\varepsilon^w, & f_p^w>0\\ > \mu^w+\varepsilon^w, & f_p^{w,*}=0\end{cases} \quad \forall p\in R_w, w\in W. \tag{7}$$

Theorem 1. *The fixed point of dynamical system coincides with BRUE path flow pattern.*

Proof: Part 1: *BRUE path flow pattern implies equilibrium of the day to day dynamical model.* From Eq. (7), if a path flow pattern **f** is a BRUE path flow pattern, we have:

$$f_p^w(c_p^w-\mu^w-\varepsilon^w)_+=0. \tag{8}$$

Then the equilibrium state can be inferred from Eq. (8) for any fixed p:

(1) If $f_p^w>0$, from Eq. (7), we have $c_p^w\le\mu^w+\varepsilon^w$. Then we have $c_p^w\le c_q^w+\varepsilon^w$, since μ^w is the minimal path cost between OD pair w. Hence $f_p^w(c_p^w-c_q^w-\varepsilon^w)_+=0$.

(2) If $f_p^w=0$, it is obviously the RHS of Eq. (4) is equal to 0.

Part 2: *The fixed point of dynamical model (4) implies that the path flow pattern is BRUE path flow pattern.* Assume that the path p has the minimum path cost μ^w between OD pair w. If the flow pattern is in equilibrium, we have

$$\frac{df_p^w}{dt}=k_w\left(\sum_{q\in R_w}\left(\frac{f_q^w}{mc_p^w}\left(c_q^w-\mu^w-\varepsilon^w\right)_+ - \frac{f_p^w}{mc_q^w}\left(\mu^w-c_q^w-\varepsilon^w\right)_+\right)\right)=0. \tag{9}$$

Given that $\left(\mu^w-c_q^w-\varepsilon^w\right)_+=0$, we have:

$$\frac{df_p^w}{dt}=k_w\left(\sum_{q\in R_w}\left(\frac{f_q^w}{mc_p^w}\left(c_q^w-\mu^w-\varepsilon^w\right)_+\right)\right)=\mathbf{0}. \tag{10}$$

All items in Eq. (10) are positive, which leads to:

$$f_q^w\left(c_q^w-\mu^w-\varepsilon^w\right)_+=0 \qquad \forall\, q\in R_w, w\in\mathbf{W}. \qquad \blacksquare$$

Next we show that the BRUE will be globally reached over time. This asymptotical property is guaranteed by revealing that the dynamical system shown in Eq. (4) satisfies the BRUE-RBAP condition in Ye and Yang (2017).

Theorem 2. *The proposed Eq. (4) satisfies RBAP-BRUE defined in Ye and Yang (2017).*

Proof:

$$\sum_{w\in W}\sum_{p\in R_w} c_p^w \frac{df_p^w}{dt} = \sum_{w\in W}\sum_{p\in R_w} c_p^w k_w \left(\sum_{q\in R_w}\left(\frac{f_q^w}{mc_p^w}\left(c_q^w - c_p^w - \varepsilon^w\right)_+ - \frac{f_p^w}{mc_q^w}\left(c_p^w - c_q^w - \varepsilon^w\right)_+\right)\right) \quad (11)$$

When transportation system is under dis-equilibrium, we know that the above formula has even number of non-zero pairs. Without loss of generality, assume that the path cost of one pair of routes which consists of path p, q connected by the same OD pair w satisfying $c_p^w > c_q^w + \varepsilon^w$. Under this scenario, it is obvious that $f_p^w > 0$. Hence,

$$\begin{aligned} &c_p^w k_w \left(\frac{f_q^w}{mc_p^w}\left(c_q^w - c_p^w - \varepsilon^w\right)_+ - \frac{f_p^w}{mc_q^w}\left(c_p^w - c_q^w - \varepsilon^w\right)_+\right) \\ &\qquad + c_q^w k_w \left(\frac{f_p^w}{mc_q^w}\left(c_p^w - c_q^w - \varepsilon^w\right)_+ - \frac{f_q^w}{mc_p^w}\left(c_q^w - c_p^w - \varepsilon^w\right)_+\right) \\ &= -c_p^w k_w \frac{f_p^w}{mc_q^w}\left(c_p^w - c_q^w - \varepsilon^w\right) + c_q^w k_w \frac{f_p^w}{mc_q^w}\left(c_p^w - c_q^w - \varepsilon^w\right). \\ &= -k_w \frac{f_p^w}{mc_q^w}\left(c_p^w - c_q^w\right)\left(c_p^w - c_q^w - \varepsilon^w\right) \end{aligned} \quad (12)$$

Given that $c_p^w > c_q^w + \varepsilon^w$,

$$-k_w \frac{f_p^w}{mc_q^w}\left(c_p^w - c_q^w\right)\left(c_p^w - c_q^w - \varepsilon^w\right) < 0 \quad (13)$$

Since the zero pairs are ineffectual, we arrive at $\sum_{w\in \mathbf{W}}\sum_{p\in R_w} c_p^w \frac{df_p^w}{dt} < 0$. ■

Theorem 3 (*Ye and Yang, 2017****).*** *Assume the BRUE-RBAP has classical solutions that are continuous functions of the initial conditions, and the link travel cost functions are separable. If $f(0) \in \Omega$, then $f(t) \to \Omega^*$ as $t \to \infty$.*

We end this section by giving the following lemma on the stability property.

Lemma. Under *The proposed dynamical system (8) with separable link cost function, if* $\mathbf{f}(0) \in \Omega$, *it holds that* $\mathbf{f}(\mathrm{t}) \in \Omega^*$ *as* $\mathrm{t} \to \infty$, *i.e., the dynamical system is globally asymptotically stable.*

Proof: It can be verified by ***Theorem 2*** and ***3*** *with assuming separable travel cost functions.*

4 Numerical Experiment

We use the classical Braess network which has one OD pair, five links, and three paths (see Figure 1). Path1: O->1->3->D; Path2: O->2->4->D; Path3: O->2->5->3->D. For each link, the travel time function is chosen as the well-known BPR function: $c_a(v_a) = c_a^0\left[1 + 0.15\left(\frac{v_a}{q_a}\right)^4\right], \forall a \in A$, where c_a^0 is the free flow travel time and q_a is the link capacity. Here, $c^0 = (15, 10, 10, 15, 5)$ and $q = (1800, 1800, 3600, 1800, 1800)$. The OD demand is fixed to be 5400. The indifference band $\varepsilon = 1$. At $t = 0$, the system starts evolving with different initial states. Numerically, if marginal cost becomes zero (link flow is zero), we will add a small positive value into the marginal term. Figure 2a compares the evolution trajectories with and without considering the marginal cost term. The figure reveals that the marginal term will increase flow swapping rate. At $t = 0$, path 1 which has less traffic and larger capacity will be more attractive due to the marginal term. Moreover, the fast swapping speed may induce a different final equilibrium state in which the path with small marginality cost will have more traffic. Figure 2b shows the flow trajectories of path 2 and 3 under the MU-BRUE BNN dynamics. With the assumption of separable link cost function, the MU-BRUE BNN dynamics can be proved to satisfy

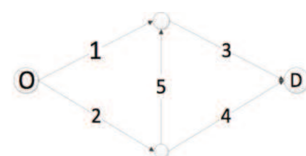

Figure 1. Network structure

the BRUE-RBAP, even though the proof is omitted here to conserve space. The figure shows that dynamic will also converge to the BRUE set which is possibly non-convex. At first the convergent speed is fast and it will gradually diminish when approaching the equilibrium.

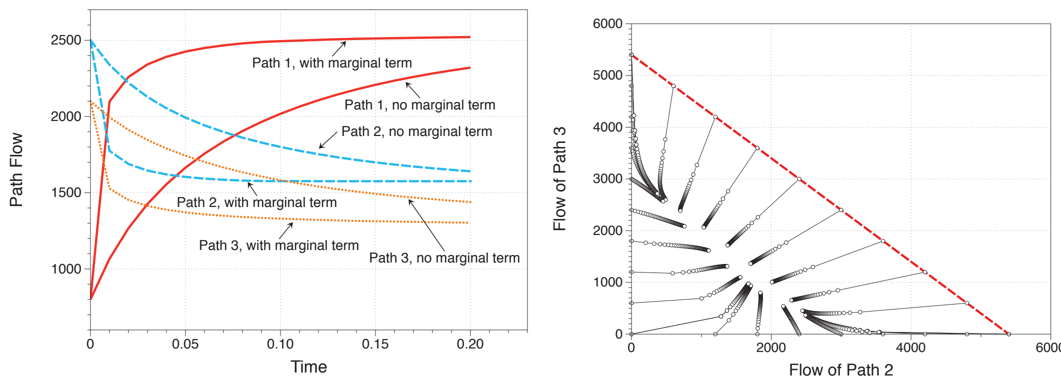

Figure 2: a. Flow evolution of MU-BRUE Smith dynamic; b. Trajectories of MU-BRUE BNN dynamic with $\epsilon = 1$

5 Conclusions

In this paper, the Smith dynamic and BNN dynamic are extended to capture both the marginality and the bounded rationality in the flow evolution process from dis-equilibrium to equilibrium. The properties of the extended Smith dynamic are theoretically investigated and the extended BNN dynamic is numerically evaluated. We show that the stationary point of the MU-BRUE Smith dynamic coincides with BRUE path flow pattern and prove this dynamic follows RBAP-BRUE. Numerical experiments show that the marginal cost term can accelerate the flow swapping speed when the system is far away from the equilibrium. And faster swapping speed may induce a different equilibrium state in which the path with smaller marginality cost will endure more traffic.

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