Proc. of the 6th Intl. Symposium on Reliability Engineering and Risk Management (6ISRERM)
31 May – 1 June 2018, Singapore
Editor(s) Xudong Qian, Sze Dai Pang, Ghim Ping Raymond Ong, Kok-Kwang Phoon

ISRERM 2018

SYSTEM RELIABILITY ANALYSIS BASED ON ADAPTIVE ESTIMATION OF STATISTICAL MOMENTS AND MAXIMUM ENTROPY

Zhengliang Li[1,2], Guo Ye[1], Wenliang Fan[1,2], Pu Deng[3]

[1]*School of Civil Engineering, Chongqing University, Chongqing, China*
[2]*Key Laboratory of New Technology for Construction of Cities in Mountain Area (Chongqing University), Ministry of Education, Chongqing, China*
lizhengl@hotmail.com
[1]*School of Civil Engineering, Chongqing University, Chongqing, China*
guoye@cqu.edu.cn
[1]*School of Civil Engineering, Chongqing University, Chongqing, China*
[2]*Key Laboratory of New Technology for Construction of Cities in Mountain Area (Chongqing University), Ministry of Education, Chongqing, China*
davidfwl@126.com
[3]*Chongqing Jianzhu College, Chongqing, China*
657050439@qq.com

The system reliability analysis is one of the main topics in the field of reliability engineering. Although progress has been made, there remain multiple challenges in the existing methods including the correlation of failure modes, the combinatorial explosion problems, and excessive computational efforts.

In this paper, a new moment method with high accuracy, efficiency is presented. Firstly, by introducing the complete system failure process (originally defined as the development process of nonlinearity), an equivalent performance function describing a structural system is obtained. Secondly, point estimate method (PEM) based on adaptive dimensional decomposition is adopted to compute the first four moments of the equivalent performance function. Thirdly, approximated probability density function (PDF) of the equivalent performance function based on the maximum entropy principle is derived to estimate the system failure probability. An example is investigated to illustrate the numerical accuracy, efficiency of the proposed method in comparison to the Monte Carlo simulation method.

Keywords: System reliability, complete system failure process, point estimate method (PEM), adaptive dimensional decomposition, maximum entropy.

1 Introduction

With the increasing awareness and importance of probability-based design methods and the development of modern computational techniques, reliability analysis has been playing an increasingly significant role in the structural engineering (Thoft-Christensen 1982, Ditlevsen 1996). In general, structural reliability analysis is usually classified into two categories: structural member reliability and structural system reliability (Thoft-Christensen 1986).

The structural system reliability considers multiple failure modes, and classical system reliability focuses on the collapse or forming mechanisms of perfectly elastoplastic and elastic-brittle structures (Melchers 1984, Thoft-Christensen 1986, Feng 1988, Zhao 2001). There usually exist multiple potential failure paths that lead to the collapse of structure, thus methods based on the dominant failure modes are developed naturally for the classical system reliability, which mainly include two steps: (i) identifying the dominant failure modes of a structure, including β-unzipping method (Thoft-Christensen 1986), branch and bound method (Thoft-Christensen 1986), truncating enumeration method (Melchers 1984), criterion methods (Feng 1988) and so on; and (ii) approximating reliability based on the identified failure modes, including the PNET (Ang 1984), the lower-upper bound method (Ditlevsen 1979, Feng 1989). Although the achievements were fruitful, this family of methods have common restrictions that: (i) the number of failure modes increases dramatically as the structure redundancy increases (Onoufriou 2001); and (ii) the correlation information between failure modes is difficult to evaluate. To avoid the restrictions of traditional system reliability methods, Chen and Li (Chen 2007) proposed the development process of nonlinearity, which is also defined as complete system failure process (Fan 2016). Based on this idea (Chen 2007, Fan 2016), the classical system reliability is effectively described by a single equivalent performance function.

The present paper is organized as follows. In Section 2, an equivalent performance function for system reliability of structure is formulated based on the complete system failure process; the moments of this performance function is evaluated by the adaptive bivariate dimensional decomposition method; and the system reliability is evaluated based on maximum entropy with the first four moments. An example is investigated in Section 3 to verify the accuracy, efficiency of the proposed method. At last some conclusions are drawn in Section 4.

2 Improved moment method for system reliability

2.1 *On the complete system failure process*

In regard to the complete system failure process (Chen 2007), the system reliability of the structure with elastic-plastic material is equivalent to the probability of its ultimate capacity being greater than the applied load, namely

$$R = \Pr\left\{P_{\max}\left(\boldsymbol{\Theta}\right) - P\left(\boldsymbol{\Theta}_L\right) > 0\right\} \tag{1}$$

where R is the reliability, $\boldsymbol{\Theta}=(\boldsymbol{\Theta}_S, \boldsymbol{\Theta}_L)=(\Theta_1,\Theta_2,\ldots,\Theta_n)^T$ is the random vector, in which $\boldsymbol{\Theta}_S$ and $\boldsymbol{\Theta}_L$ are random vectors of the structural parameters and loads, respectively, $P_{\max}(\cdot)$ is the ultimate limit capacity corresponding to the load parameter P. Let

$$Z_{\max}\left(\boldsymbol{\Theta}\right) = P_{\max}\left(\boldsymbol{\Theta}\right) - P\left(\boldsymbol{\Theta}_L\right) \tag{2}$$

in which $Z_{\max}(\boldsymbol{\Theta})$ denotes the margin for ultimate limit capacity of the structure. Then, Eq. (1) becomes

$$R = \Pr\left\{Z_{\max}\left(\boldsymbol{\Theta}\right) > 0\right\} \tag{3}$$

In contrast to the traditional system reliability that requires identifying the dominant failure modes, there is only one equivalent failure mode based on the complete system failure process as summarized above.

2.2 *Equivalent performance function remodeled by Nataf transformation*

By introducing the Nataf transformation (Li 2008) into the performance function $Z(\boldsymbol{\Theta})$, it can be inferred as

$$Z_{\max}(\boldsymbol{\Theta}) = Z_{\max}\left(T^{-1}\left(U_1,\cdots,U_i,\cdots U_n\right)\right) = H(\boldsymbol{U}) \tag{4}$$

where $T^{-1}(\cdot)$ is the inverse transform of the Nataf transformation $T(\cdot)$, and $\boldsymbol{U}=\{U_1,\ldots,U_n\}$ is a mutually independent standard normal vector.

2.3 *Point estimate for moments based on adaptive bivariate dimensional decomposition*

According to Fan and Wei (2015), the mean value and the first q central moments ($q \geq 2$) of Z are as follows

$$\mu_Z = E[Z] \approx \sum_{i_1<i_2} E\left[H\left(U_{i_1},U_{i_2},\underline{\boldsymbol{u}}_{i_1i_2,c}\right)\right] - (N-2)\sum_{k=1}^{N} E\left[H\left(U_k,\underline{\boldsymbol{u}}_{k,c}\right)\right] + \frac{(N-1)(N-2)}{2}H(\boldsymbol{u}_c) \tag{5a}$$

$$\begin{aligned} M_{Z,q} &= E\left[(Z-\mu_Z)^q\right] \approx \\ &\sum_{i_1<i_2} E\left[\left(H\left(U_{i_1},U_{i_2},\underline{\boldsymbol{u}}_{i_1i_2,c}\right)-\mu_Z\right)^q\right] - (N-2)\sum_{k=1}^{N} E\left[\left(H\left(U_k,\underline{\boldsymbol{u}}_{k,c}\right)-\mu_Z\right)^q\right] + \frac{(N-1)(N-2)}{2}\left(H(\boldsymbol{u}_c)-\mu_Z\right)^q \end{aligned} \tag{5b}$$

where $E[.]$ is the expectation operator, μ_Z is the mean value of Z, $\boldsymbol{u}_c$ is the reference point, and $\underline{\boldsymbol{u}}_{i_1i_2,c}$ is a sub-vector of $\boldsymbol{u}_c$ without the corresponding coordinates of U_{i_1}, U_{i_2}.

Introducing the delineation of the cross terms (Fan 2015) of the cross terms of U_i and U_j, Then $E\left[\left(H\left(U_i,U_j,\underline{\boldsymbol{u}}_{ij,c}\right)-\mu_Z\right)^q\right]$ becomes

$$\begin{aligned} &E\left[\left(H\left(U_i,U_j,\underline{\boldsymbol{u}}_{ij,c}\right)-\mu_Z\right)^q\right] = \\ &E\left[\left(I_h\left(U_i,U_j\right)H\left(U_i,U_j,\underline{\boldsymbol{u}}_{ij,c}\right) + \left(1-I_h\left(U_i,U_j\right)\right)\left(H\left(U_i,\underline{\boldsymbol{u}}_{i,c}\right)+H\left(U_j,\underline{\boldsymbol{u}}_{j,c}\right)-H(\boldsymbol{u}_c)\right)-\mu_Z\right)^q\right] \end{aligned} \tag{6}$$

where $I_h\left(U_i,U_j\right)$ is the indication function defined as

$$I_h\left(U_i,U_j\right) = \begin{cases} 1, & \text{if the cross terms of } U_i \text{ and } U_j \text{ exist} \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

If $I_h\left(U_i,U_j\right)=1$, then Eq. (6) becomes

$$\begin{aligned} &E\left[\left(I_h\left(U_i,U_j\right)H\left(U_i,U_j,\underline{\boldsymbol{u}}_{ij,c}\right) + \left(1-I_h\left(U_i,U_j\right)\right)\left(H\left(U_i,\underline{\boldsymbol{u}}_{i,c}\right)+H\left(U_j,\underline{\boldsymbol{u}}_{j,c}\right)-H(\boldsymbol{u}_c)\right)-\mu_Z\right)^q\right] \\ &= E\left[\left(H\left(U_i,U_j,\underline{\boldsymbol{u}}_{ij,c}\right)-\mu_Z\right)^q\right] \end{aligned} \tag{8}$$

If $I_h\left(U_i,U_j\right)=0$, then Eq. (6) becomes

$$
\begin{aligned}
&E\left[\left(I_h\left(U_i,U_j\right)H\left(U_i,U_j,\underline{\boldsymbol{u}}_{ij,c}\right)+\left(1-I_h\left(U_i,U_j\right)\right)\left(H\left(U_i,\underline{\boldsymbol{u}}_{i,c}\right)+H\left(U_j,\underline{\boldsymbol{u}}_{j,c}\right)-H\left(\boldsymbol{u}_c\right)\right)-\mu_Z\right)^q\right] \\
&=E\left[\left(\left(H\left(U_i,\underline{\boldsymbol{u}}_{i,c}\right)-\mu_Z\right)+\left(H\left(U_j,\underline{\boldsymbol{u}}_{j,c}\right)-\mu_Z\right)-\left(H\left(\boldsymbol{u}_c\right)-\mu_Z\right)\right)^q\right] \\
&=\sum_{k_1+k_2+k_3=q}\frac{q!}{k_1!k_2!k_3!}\left(\mu_Z-H\left(\boldsymbol{u}_c\right)\right)^{k_3}E\left[\left(H\left(U_i,\underline{\boldsymbol{u}}_{i,c}\right)-\mu_Z\right)^{k_1}\right]E\left[\left(H\left(U_j,\underline{\boldsymbol{u}}_{j,c}\right)-\mu_Z\right)^{k_2}\right]
\end{aligned}
\tag{9}
$$

It is worth pointing out that: If $I_h\left(U_i,U_j\right)=0$, then the Eq. (6) only involves one dimensional integrals. Obviously, introducing the delineation of the cross terms improves the efficiency of the moments estimation for the response function with $I_h\left(U_i,U_j\right)=0$.

Eq. (5a) and Eq. (5b) are only expectation operation of functions with independent standard normal variables, and the Gauss-Hermite quadrature is useful, namely

$$
\begin{cases}
E\left[\left(H\left(U_k,\underline{\boldsymbol{u}}_{k,c}\right)-C\right)^q\right]=\sum_{l_0=1}^{l}\frac{w_{GH,l_0}}{\sqrt{\pi}}\left(H\left(\sqrt{2}x_{GH,l_0},\underline{\boldsymbol{u}}_{k,c}\right)-C\right)^q \\
E\left[\left(H\left(U_i,U_j,\underline{\boldsymbol{u}}_{ij,c}\right)-C\right)^q\right]=\sum_{l_1=1}^{l}\sum_{l_2=1}^{l}\frac{w_{GH,l_1}w_{GH,l_2}}{\pi}\left(H\left(\sqrt{2}x_{GH,l_1},\sqrt{2}x_{GH,l_2},\underline{\boldsymbol{u}}_{ij,c}\right)-C\right)^q
\end{cases}
\tag{10}
$$

where $x_{GH,p}$ and $w_{GH,p}$ are the abscissas (in ascending order) and weights of the Gauss–Hermite quadrature formula as illustrated in http://keisan.casio.com/exec/system/1330940731.

2.4 *System reliability analysis by maximum entropy*

The information-theoretic entropy of a continuous random variable Y with PDF $f_Y(\mathrm{y})$ is defined as

$$H[f_Y]=-\int_Y f_Y(y)\log[f_Y(y)]dy \tag{11}$$

Given m moments of Y, an estimation $\hat{f}_Y(\mathrm{y})$ of the true PDF $f_Y(\mathrm{y})$ can be obtained by the principle of maximum entropy (MaxEnt) in the following way (Zhang 2013):

Find: $f_Y y$

Maximize: $H[f_Y]=-\int_Y f_Y(y)\log[f_Y(y)]dy$

Subject to: $\int_Y y^{\alpha_k} f_Y(y)dy=M_Y^{\alpha_k}$ (for k=1,2,...,m) (12)

where $M_{Y\alpha_k}$ is an $\alpha_k th$ order origin moment of Y.

The estimated PDF has the following generic form:

$$\hat{f}_Y(y)=\exp(-\sum_{k=0}^{m}\lambda_k y^{\alpha_k}) \tag{13}$$

where

$\alpha_0=0$, and $\lambda_0=\log[\int_Y \exp(-\sum_{k=1}^{m}\lambda_k y^{\alpha_k})dy]$

λ and α of the MaxEnt distribution can be carried out in MATLAB with the simplex search method (Lagarias 1998).

3 Numerical examples

In this example, consider a ten-bar truss shown in Fig. 1. The geometric dimensions of this truss are l_0=1.0m, and h=1.5m; the section areas of all bars are 0.01m^2. The material is assumed to be elastic perfectly plastic with the elastic modulus of E=2.06×10^{5}MPa. There are four lognormal random variables involved, namely the load F_1 and the yield stresses of the bars σ_i (i=1,2,3). The mean value of F_1 is 500 KN, and its coefficient of variation (cov) is assumed to be 0.2. The statistical information of the yield stress σ_i is listed in Table 1.

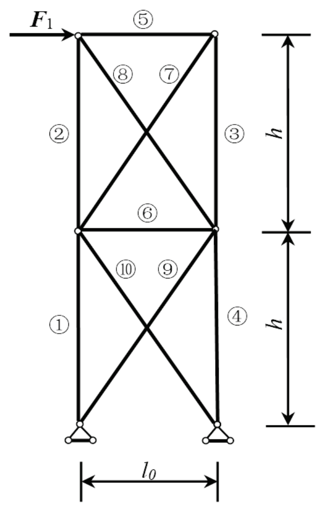

Fig.1 Ten-bar elastoplastic truss

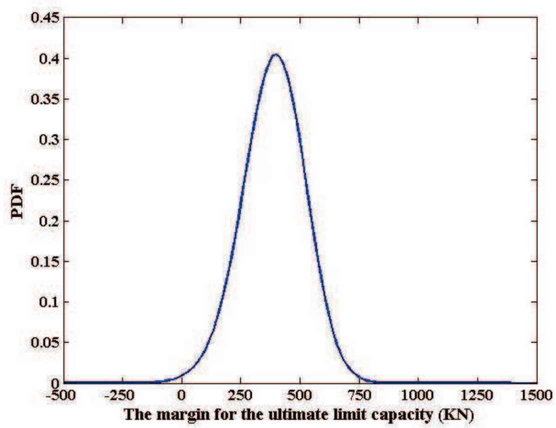

Fig.2 Distribution of the margin for the ultimate capacity

Table 1 The statistical information of yielding stresses of bars

	Member	Mean value(MPa)	cov
σ1	①, ④	200	0.1
σ2	②, ③, ⑤, ⑥	200	0.1
σ3	others	200	0.1

According to the complete system failure process, the system reliability of this truss can be expressed equivalently by

$$R = \Pr\left\{P_{\max}\left(\sigma_1, \sigma_2, \sigma_3, F_1\right) - P\left(F_1\right) > 0\right\} = \Pr\left\{Z_{\max}\left(\sigma_1, \sigma_2, \sigma_3, F_1\right) > 0\right\} \tag{14}$$

If σ_i (i=1,2,3) and F_1 are assumed to be mutually independent, the failure probability from Monte Carlo simulation with 1 million samples is 3.671×10^{-3}. Through the proposed method, the first four moments of the margin for ultimate limit capacity are approximately $\mu_{Z\max}$=387.66, $\sigma_{Z\max}$=133.41, $\alpha_{3,Z\max}$=-0.1937, $\alpha_{4,Z\max}$=3.2535, and the probability density function (PDF) of the equivalent performance function is obtained as illustrated in Fig. 2. The reliability index is 2.675 with the corresponding failure probability of 3.734×10^{-3}. This result agrees well with the result of the Monte Carlo method.

4 Conclusions

In this work, by introducing the complete system failure process to obtain the equivalent performance function, and then combining the adaptive PEM with the maximum entropy to calculate the failure probability, an improved moment method for system reliability analysis is proposed. An numerical example is presented to verify the effectiveness of the proposed method.

The following conclusions can be drawn:

(i) By introducing adaptive DRM into evaluation of statistical moments, the efficiency is improved.

(ii) The proposed method, which shows good efficiency, accuracy, provides a complete systematic solution to system reliability analysis, and avoids the common restrictions of existing methods.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No. 51478064 and No. 51678092), and NSFC-JSPS China-Japan Scientific Cooperation Project (NSFC Grant No. 51611140123). These supports are gratefully acknowledged

References

Ang AH-S, Tang WH. *Probability concepts in engineering planning and design*. John Wiley & Sons, 1984.

Chen JB, Li J. Development-process-of-nonlinearity-based reliability evaluation of structures. *Probabilistic Eng Mech* , 22(3): 267-275, 2007.

Ditlevsen O, Madsen HO. *Structural reliability methods*. Chichester: John Wiley& Sons Ltd. 1996.

Ditlevsen O. Narrow reliability bounds for structural systems. *Struct Eng Mech* , 7 (4): 453–525, 1979.

Fan WL, Wei JH, Ang AH-S, Li ZL. Adaptive estimation of statistical moments of the responses of random systems. *Probabilistic Eng Mech* , 43: 50-67, 2015.

Feng YS. Enumerating significant failure modes of structural system by using criterion methods. *Comput Struct* , 30 (3): 1153–1157, 1988.

Feng YS. A method for computing structural system reliability with high accuracy. *Comput Struct* , 339 (1): 1–5, 1989.

Hu Z, Mahadevan S. Time-dependent system reliability analysis using random field discretization. *J Mech Des* , 137(10): 101404-1–101404-10, 2015.

Lagarias J, Reedss J, Wright M, Wright P. Convergence properties of the Nelder-Mead simplex method in low dimensions. *SIAM J Optim* , 9(1):47-112, 1998.

Li HS, Lv ZZ, Yuan XK. Nataf transformation based points estimate method. *Chinese Science Bulletin* , 53(17): 2586-2592, 2008.

Mahadevan S, Hu Z. Resilience assessment based on time-dependent system reliability analysis. *J Mech Des ,* 138(11): 111404-1–111404-11, 2016.

Melchers RE, Tang LK. Dominant failure modes in stochastic structural systems. *Struct Saf* , 2 (2): 127–143, 1984.

Onoufriou T, Forbes V J. Developments in structural system reliability assessments of fixed steel offshore platforms. *Reliab Eng Syst Saf ,* 71(2):189-199, 2001.

Thoft-Christensen P, Baker MJ. *Structural reliability theory and its applications.* Springer-Verlag. 1982.

Thoft-Christensen P, Murotsu Y. *Application of structural systems reliability theory.* Springer-Verlag. 1986.

Zhang X, Pandey M. Structural reliability analysis based on the concepts of entropy, fractional moment and dimensional reduction method. *Struct Saf* , 43 (9): 28–40, 2013.

Zhao YG, Ang AH-S. System reliability assessment by method of moments. Struct Eng , 129 (10): 1341–1349, 2001.