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MOBILIZED YOUNG'S MODULUS FOR A FOOTING

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This study investigates the possibility of representing the mobilized Young's modulus (E_m) for a footing in a spatially varying medium - the Young's modulus actually "felt" by the footing - using a spatial average. The E_m is simulated by a homogenization procedure that matches the responses between a random finite element analysis (RFEA) and a homogeneous finite element analysis. Emphasis is placed on whether the spatial average can well represent the numerical value of E_m in each spatially varying realization. It is found that the conventional spatial averaging model that treats all soil regions equally important in general cannot satisfactorily represent E_m. Numerical results show that the concept of "mobilization" is essential: highly mobilized soil regions close to the footing should be given larger weights than non-mobilized remote regions. A key contribution of this paper is the development of a simple method to estimate the non-uniform weights for the spatial averaging using a single run of a homogeneous finite element analysis.

Keywords: spatial variability, Young's modulus, homogenization, footing.

1 Introduction

The spatial variability of soil parameters has profound impact to the behavior of a geotechnical structure. For footings on soils with isotropic scales of fluctuation (SOF), an important observation made in Fenton and Griffiths (2002, 2005) is that the statistics (mean and variance) for the "mobilized" Young's modulus (E_m) are similar to those for the geometric average (E_g) over a prescribed domain under the footing. Note that the similarity in the statistics does not imply a very strong correlation. For the footing problem, there is a significant scatter between E_m and E_g. The purpose of this study is to propose a new spatial averaging method for the footing problem so that not only E_m and the spatial average have similar statistics but they are also very strongly correlated. It will be clear that the resulting spatial average is not a uniform mobilization but a non-uniform mobilization. The soil elements significantly influenced by the footing load are highly mobilized, whereas those remote to the footing have negligible mobilization. More importantly, it is found that the degree of mobilization can be well quantified by certain physical quantity that is derived from the stress/strain change due to the footing load, and the spatial distribution of such a physical quantity can be obtained by a single run of a deterministic finite element analysis (FEA).

2 The Footing Problem

Consider a footing on a two-dimensional (2D) spatially variable soil mass, modeled by finite elements (FE) as shown in Figure 1. The spatially variable Young's modulus, denoted by E(x,z), is modeled as a stationary lognormal random field with inherent mean = μ = 20,000 kN/m^2 and inherent coefficient of variation (COV) = V = 1.0, with auto-correlation structure defined by the single exponential model (Vanmarcke 1977). The scale of fluctuation (SOF) is denoted by δ. Only isotropic random fields are considered, i.e., the horizontal and vertical scales of fluctuation are equal. Four SOFs are considered: δ = 1m, 2m, 5m, and 10m (δ/B = 0.5, 1, 2.5, and 5). For each δ, one thousand realizations of E random fields are simulated. The Poisson's ratio (ν) is assumed to be constant (ν = 0.3), because the impact of the spatial variability of the Poisson's ratio is insignificant.

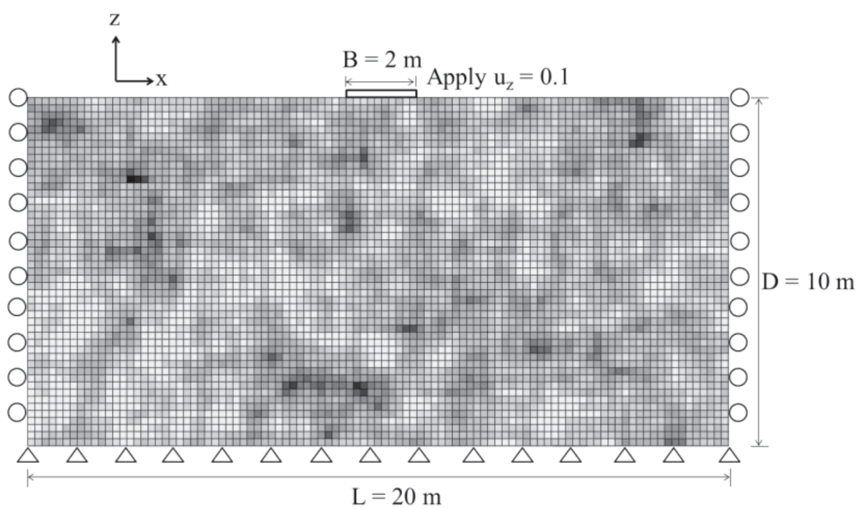

Figure 1 Realization of the E random field for the 2D footing problem with $\delta_x = \delta_z$ = 1 m.

For each random field realization, a geostatic step is adopted to build up the in-situ stress field over the entire soil mass. Then, the footing is loaded with a vertical downward uniform displacement of 0.1 m in the FE simulation, not allowing any rotations. The resulting total contact force between the footing and the soil mass is recorded. Another FE simulation with homogeneous E is conducted, following the same geostatic step and the same displacement-controlled loading. The homogeneous E value is adjusted until the total contact force matches that for the random field realization. The adjusted E value is called the mobilized Young's modulus, E_m, for the random field realization.

The 20m×10m plane strain rectangular domain is modeled by the FE mesh shown in Figure 1. Each FE is a 4-noded element of size = 0.2m×0.2m. Each FE follows an isotropic elasticity model with E = its local geometric average, ν = 0.3, and unit weight γ = 20 kN/m^3. The boundary conditions for the FEA are also shown in Figure 1. The footing is assumed to be rigid and the soil-footing interface is assumed to be rough. The Young's modulus of the soil mass is modeled as a stationary lognormal random field with inherent mean = μ = 20,000 kN/m^2 and inherent coefficient of variation V = 1.0. Cases with $\delta_x = \delta_z = \delta$ will be first considered. Five SOFs are considered: δ = 1m, 2m, 5m, 10m, 100m, and 1000m (δ/B = 0.5, 1, 2.5, 5, 50, and 500). For each δ, one thousand realizations of E random fields are simulated. Figure 2 shows the pairwise plot for the simulated E_m/μ versus E_g/μ, where the geometric average E_g is taken over the 1B×5B domain under the footing. This averaging domain was considered in Fenton and Griffiths (2002). Although the statistics (e.g., mean and COV) of E_m and E_g are similar: the correlation coefficient between them is not very strong, e.g., the Pearson correlation coefficient $\rho = 0.68$ for $\delta/B = 0.5$.

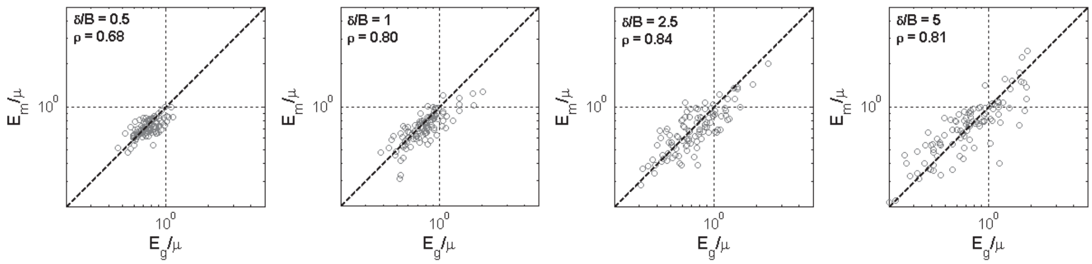

Figure 2 E_m/μ versus E_g/μ relationships.

3 New Spatial Averaging Method

One possible explanation for why E_m cannot be well represented by E_g is that elements are not mobilized uniformly. The governing volume primarily lies below the footing, and the soil volume remote from the footing is not mobilized. The non-uniform mobilization is quantified by unequal weights that are to be calibrated by random finite element analysis (RFEA) results. There are one thousand realizations of RFEA for each SOF, producing numerous calibration cases. For the k-th calibration case, there is a realization of $E_{m,k}$ and a realization of ($E_{1,k}$, $E_{2,k}$, …, $E_{n,k}$), where $E_{i,k}$ is the Young's modulus assigned to the i-th finite element for the k-th calibration case, and n is the total number of calibration cases. The following linear regression in the form of weighted geometric average is adopted:

$$\ln\left(E_{m,k}\right) \approx \sum_i w_i \times \ln\left(E_{i,k}\right) \triangleq \ln\left(E_{wg,k}\right) \tag{1.)}$$

where E_{wg} denotes the weighted geometric average; w_i is the weight for the i-th element that quantifies the degree of mobilization for the i-th element. To suppress the over-fit, the regularized least square (RLS) method (e.g., Tikhonov regularization; see Tikhonov and Arsenin 1977) is adopted to determine the unknown weights **w** by introducing a penalty term to encourage a "regularized" **w** solution. The RLS method is to minimize the following objective function:

$$\min_{\mathbf{w}} \sum_k e_k^2 + \lambda \times \sum_i w_i^2 \tag{2.)}$$

where $e_k = \ln(E_{m,k}) - \Sigma_i[w_i \times \ln(E_{i,k})]$ is the error for the k-th calibration case; the penalty term $\lambda \times \Sigma_i w_i^2$ discourages large weights; the parameter λ is called the Tikhonov factor. Golub et al. (1979) showed that the optimal λ that minimizes the leave-one-out cross-validation error. For our case, the optimal λ (λ^*) is 3845. The solution w^* has analytical solution because the objective function is a quadratic function of **w**. Figure 3 shows the grey scale plot for the resulting $\mathbf{w}^*$. The sum of all optimal weights is 0.9952, very close to unity. It is interesting to note that elements remote to the footing have negligible weights. It is found that $\mathbf{w}^*$ not only provides excellent fit to the calibration cases but also provides satisfactory prediction to the independent validation cases. Figure 4 shows the pairwise plot for E_m/μ versus E_{wg}/μ for extra validation cases.

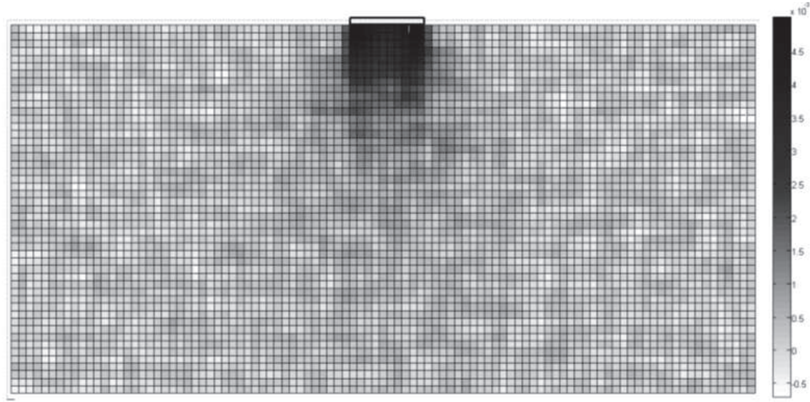

Figure 3 Grey scale plot for $\mathbf{w}^*$.

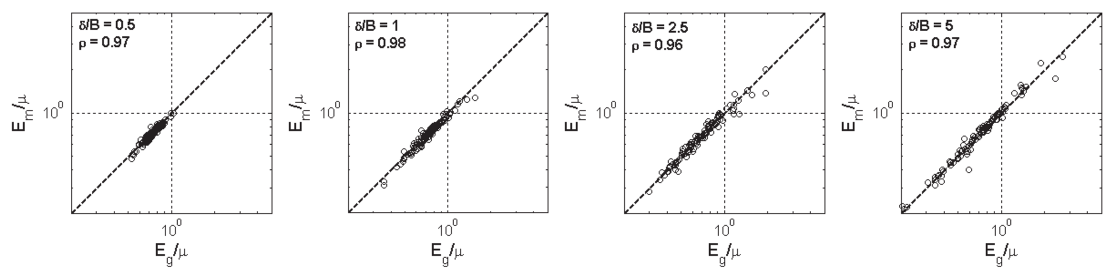

Figure 4 E_m/μ versus E_{wg}/μ relationships for extra validation cases.

3.1 *Correlation to stress/strain factors*

It is expected that $\mathbf{w}^*$ should be correlated to factors such as stress/strain increments due to the footing load. A *deterministic* FEA is conducted to obtain the factors for all elements, and the correlations between the factors and $\mathbf{w}^*$ are evaluated. Let ($\Delta\sigma_{x,i}$, $\Delta\sigma_{y,i}$, $\Delta\sigma_{z,i}$, $\Delta\tau_{xz,i}$) and ($\Delta\varepsilon_{x,i}$, $\Delta\varepsilon_{z,i}$, $\Delta\varepsilon_{xz,i}$) (y is the out-of-plane direction) be the stress and strain increments of the i-th element, respectively, due to the footing load. For instance, the increment in σ_x is denoted by $\Delta\sigma_x$ = (σ_x after the footing load) – (σ_x before the footing load), and similar for other stress/strain increments. It is found that the following factor is strongly correlated to $\mathbf{w}^*$:

$$\Delta U_i = \Delta\sigma_{x,i}\Delta\varepsilon_{x,i} + \Delta\sigma_{z,i}\Delta\varepsilon_{z,i} + 2\Delta\tau_{xz,i}\Delta\varepsilon_{xz,i} \qquad (3.)$$

We termed ΔU as "pseudo incremental energy". It can be considered as the "mobilization factor" that quantifies the degree of mobilization. The pseudo incremental energies for all elements are further normalized such that they sum up to unity. The normalized pseudo incremental energy is denoted by ΔU_n. Figure 5 shows the correlation plot between ΔU_n and $\mathbf{w}^*$.

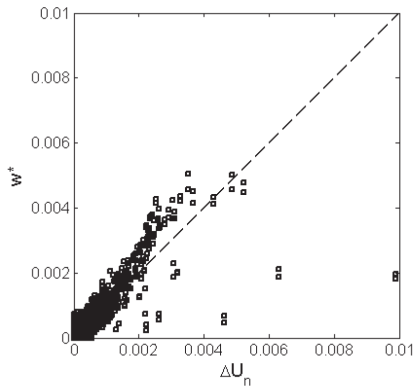

Figure 5 Correlation plot between ΔU_n and $\mathbf{w}^*$.

3.2 *Simplified procedure of simulating E_{wg}*

The following simplified procedure, called the pseudo incremental energy model, is proposed to simulate the samples of the weighted geometric average, E_{wg}. It will be shown that the resulting E_{wg} is very strongly correlated to the mobilized Young's modulus, E_m.

(i) Perform *one* deterministic FEA for the footing problem. Compute ΔU for all elements using Eq. (3), then compute the normalized form ΔU_n so that all ΔU_n sum up to unity.
(ii) Simulate the Young's moduli for all elements using random field (no need to run RFEA).
(iii) Compute the following weighted geometric average E_{wg}:

$$\ln\left(E_{wg}\right) = \sum_{i} \Delta U_{n,i} \times \ln\left(E_i\right) \tag{4}$$

where E_i is the Young's modulus assigned to the i-th element; $\Delta U_{n,i}$ is the normalized weight for the i-th element.

Steps (ii) and (iii) can be repeated to obtain a different sample of E_{wg} corresponding to a different random field realization. The set of weights in Eq. (4) ($\Delta U_{n,i}$) computed from a single deterministic FEA is independent of the random field realization. The practical benefit for the pseudo incremental energy model is obvious: it will be possible to simplify a RFEA problem to a weighted spatial averaging problem which is less costly and perhaps more importantly, make probabilistic design more accessible to engineers.

The weighted geometric average E_{wg} simulated by the pseudo incremental energy model can satisfactorily represent E_m for the footing problem. Consider the same footing problem in Figure 1. Figure 6 shows the pairwise plot for E_m/μ versus E_{wg}/μ, where E_{wg} is now simulated by the pseudo incremental energy model. Figure 6 can be compared with Figure 4: the former is based on the pseudo incremental energy model, whereas the latter is based on $\mathbf{w}^*$. The performance for the pseudo incremental energy model (Figure 6) is significantly better than that for the E_g model (Figure 2), although the calculation of E_g is slightly cheaper than E_{wg}.

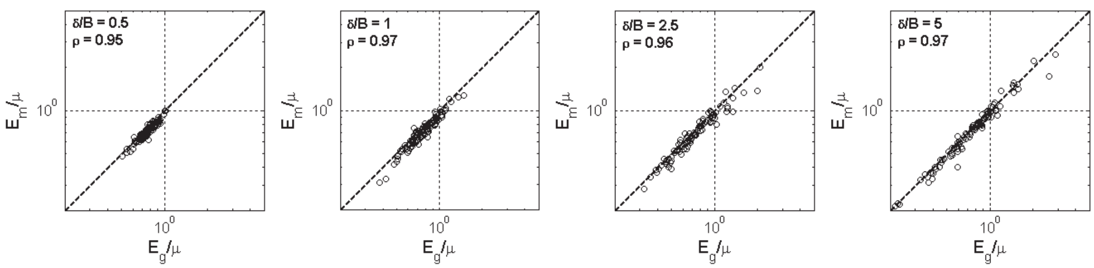

Figure 6 E_m/μ versus E_{wg}/μ relationships (E_{wg} simulated by the pseudo incremental energy model).

4 Conclusion

This study investigates the possibility of representing the mobilized Young's modulus (E_m) for a footing supported on a spatially variable medium using a suitable spatial average that is strongly correlated to E_m. It is found that the conventional spatial averaging that treats all soil regions equally important cannot satisfactorily represent E_m. Numerical evidences show that the concept of "non-uniform mobilization" is essential: highly mobilized soil regions are more important than non-mobilized regions. A key contribution in this study is to propose a simple model that can simulate the weighted spatial average that is very highly correlated to E_m. It is remarkable that the set of weights computed from the above homogeneous FEA is independent of the random field realization.

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