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UNCERTAIN STRUCTURAL ANALYSIS WITH MULTI-IMPRECISE RANDOM AND INTERVAL FIELDS

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Structural analysis involving hybrid uncertain parameters is investigated in this paper. Both multi-imprecise random fields, as well as the interval fields, are simultaneously incorporated. A new robust computational method, namely the extended unified interval stochastic sampling (X-UISS) method, is proposed for the uncertain structural analysis, such that the PDFs and CDFs of the extreme structural responses can be established.

Keywords: Imprecise random fields, interval field, hybrid uncertainty analysis.

1 Introduction

In this study, static analysis of engineering structures involving mixture of stochastic and non-stochastic uncertain system inputs is investigated. Unlike the traditional hybrid uncertain structural analysis involving random and interval variables, the concept of multi-imprecise random and interval fields has been implemented to model the spatially dependent system uncertainties that have been rapidly emerging in general engineering applications.

A new computational approach, namely the extended unified interval stochastic sampling (X-UISS) method, is specifically proposed to determine the statistical characteristics (i.e., mean and standard deviation) of the extremities (i.e., upper and lower bounds) of the concerned structural responses (i.e., displacement and stress) in the first step. Subsequently, by utilizing either parametric or non-parametric statistical inference techniques, the probability density functions (PDFs) and cumulative distribution functions (CDFs) of the extreme bounds of the structural outputs can be robustly established. Therefore, the upper and lower bounds of the structural reliability can be effectively and efficiently secured from the associated CDFs.

The applicability and effectiveness of the proposed X-UISS approach for engineering structure involving multi-imprecise random and interval fields are thoroughly demonstrated through the presentation on one numerical investigation.

2 The Concept of Imprecise Random Fields

Regarding a probability space $(\boldsymbol{\Omega}, \boldsymbol{F}, \boldsymbol{P})$, introducing $\omega \in \boldsymbol{\Omega}$, which represents a generic sampling point in the sample space $\boldsymbol{\Omega}$, $\boldsymbol{F}$ represents the σ-algebra, and $\boldsymbol{P}$ represents the measurement of the probability.

For the purpose of this paper, the imprecise random field with interval mean (i.e., $W_{\mu^I}(\mathbf{x},\omega)$) is considered throughout this study. By further implementing the Karhunen-Loève (K-L) expansion approach (Do et al 2016), the imprecise random field with bounded mean can be formulated as:

$$W_{\mu^I}(\mathbf{x},\omega)=[\underline{\mu},\overline{\mu}]+\sum_{i=1}^{\infty}\sqrt{\lambda_i}\psi_i(\omega)\phi_i(\mathbf{x}) \tag{1}$$

where $\underline{\mu},\overline{\mu}$ denotes the lower and upper bounds of the mean of the random field respectively; λ_i and $\phi_i(\mathbf{x})$ denote the ith eigenvalue and eigenfunction of the covariance function respectively; $\{\psi_i(\omega), i=1,2,\ldots\}$ denotes a set of random variables. By adopting a Kth order truncation, the Kth order approximation of the imprecise random field with bounded mean can be expressed as:

$$W_{\mu^I}(\mathbf{x},\omega)\approx\hat{W}^K_{\mu^I}(\mathbf{x},\omega)=[\underline{\mu},\overline{\mu}]+\sum_{i=1}^{K}\sqrt{\lambda_i}\psi_i(\omega)\phi_i(\mathbf{x}) \tag{2}$$

3 The Concept of Interval Fields

The concept of interval field was firstly introduced in (Verhaeghe et al 2013) and subsequently developed in (Sofi and Muscolino 2015) to model the spatially dependent uncertain parameters when the stochastic approach is prohibited. In order to present the concept of interval field in a more appropriate format, the following definitions are introduced (Wu and Gao 2017).

Definition 1. An interval field $V(\boldsymbol{\xi})$ is a collection of interval variables indexed by a continuous parameter $\boldsymbol{\xi}\in\boldsymbol{\Theta}$, where $\boldsymbol{\Theta}$ is a set of $\mathbb{R}^n$.

Definition 2. The upper bound function, denoted as $\overline{V}(\boldsymbol{\xi})$: $\mathbb{R}^n\longrightarrow\mathbb{R}$, such that $\forall\boldsymbol{\xi}_p\in\boldsymbol{\Theta}$, $V(\boldsymbol{\xi}_p)\le\overline{V}(\boldsymbol{\xi}_p)$.

Definition 3. The lower bound function, denoted as $\underline{V}(\boldsymbol{\xi})$: $\mathbb{R}^n\longrightarrow\mathbb{R}$, such that $\forall\boldsymbol{\xi}_p\in\boldsymbol{\Theta}$, $\underline{V}(\boldsymbol{\xi}_p)\le V(\boldsymbol{\xi}_p)$.

Definition 4. The mid-point function, denoted as $V^M(\boldsymbol{\xi})$: $\mathbb{R}^n\longrightarrow\mathbb{R}$, such that

$$V^M(\boldsymbol{\xi}):=\frac{\overline{V}(\boldsymbol{\xi})+\underline{V}(\boldsymbol{\xi})}{2},\ \forall\boldsymbol{\xi}\in\Theta \tag{3}$$

Definition 5. The half-width function, denoted as $V^W(\boldsymbol{\xi})$: $\mathbb{R}^n\longrightarrow\mathbb{R}$, such that

$$V^W(\boldsymbol{\xi}):=\frac{\overline{V}(\boldsymbol{\xi})-\underline{V}(\boldsymbol{\xi})}{2},\ \forall\boldsymbol{\xi}\in\Theta \tag{4}$$

Definition 6. A uniform interval field is defined such that the lower and upper bound functions are constants for all $\boldsymbol{\xi}\in\boldsymbol{\Theta}$. That is, $\underline{V}(\boldsymbol{\xi})=\underline{V}^*$ and $\overline{V}(\boldsymbol{\xi})=\overline{V}^*$, where $\underline{V}^*\le\overline{V}^*\in\mathbb{R}$.

According to **Definition 6**, the traditional interval variable can be alternative understood as a special circumstance of an interval field.

Within the context of this study, the spatial average method [5] is implemented to discretize the interval fields. That is:

$$\hat{V}(\xi) := \frac{\int_{\Psi_i} V(\xi)\, d\Psi_i}{|\Psi_i|} = \hat{V}_i,\ \xi \in \mathbf{\Psi}_i \tag{5}$$

where $\hat{V}(\boldsymbol{\xi})$ denotes the discretized interval field; $\mathbf{\Psi}_i$ denotes the domain of the ith structural element. Similarly, the lower and upper bound functions can be also discretized into two vectors $\underline{\mathbf{V}} = [\underline{V}_1, \dots, \underline{V}_n]^T$ and $\overline{\mathbf{V}} = [\overline{V}_1, \dots, \overline{V}_n]^T$ such that:

$$\mathbf{V} \in \widetilde{\Omega} := \{\mathbf{V} \in \Re^n \mid \underline{V}_p \le V_p \le \overline{V}_p,\ p = 1, \dots, n\} \tag{6}$$

By adopting the spatial average discretization method, all concerned interval fields are transformed into interval vectors with explicit upper and lower bound information.

4 The Extended Unified Interval Stochastic Sampling (X-UISS) method

By considering the abovementioned two uncertainty models simultaneously within the framework of linear static analysis, the governing equation for the hybrid uncertain static analysis of structures involving spatially dependent uncertainties can be formulated as:

Find $\mathbf{u}$ and $\boldsymbol{\sigma}$

such that:

$$\begin{cases} \mathbf{C}^T\mathbf{q} = \mathbf{F}^{RI} \\ \mathbf{C}\mathbf{u} = \mathbf{e} \\ \mathbf{S}_{\mathbf{e}}(\dot{\mathbf{E}}^{RI})\mathbf{q} + \mathbf{S}_{\mathbf{v}}(\dot{\mathbf{E}}^{RI}, \mathbf{v}^{RI})\mathbf{q} = \mathbf{e} \\ \overline{\mathbf{C}}\mathbf{q} = \boldsymbol{\sigma} \\ \dot{\mathbf{E}}^{RI} \in \{W_{\mu^I_{\dot{\mathbf{E}}}}(\mathbf{x}, \omega_{\dot{\mathbf{E}}}) \vee V_{\dot{\mathbf{E}}}(\boldsymbol{\xi})\} \\ \mathbf{v}^{RI} \in \{W_{\mu^I_{\mathbf{v}}}(\mathbf{y}, \omega_{\mathbf{v}}) \vee V_{\mathbf{v}}(\boldsymbol{\tau})\} \\ \mathbf{F}^{RI} \in \{W_{\mu^I_{\mathbf{F}}}(\mathbf{z}, \omega_{\mathbf{F}}) \vee V_{\mathbf{F}}(\boldsymbol{\gamma})\} \end{cases} \tag{7}$$

where $\mathbf{C} \in \mathbb{R}^{\tilde{n}\times d}$ is the overall compatibility matrix, and the transpose of the compatibility matrix $\mathbf{C}^T \in \mathbb{R}^{d\times\tilde{n}}$ is the global equilibrium matrix; d represents the number of degree of freedom; $\tilde{n}$ collects the number of independent-force systems across the entire structural domain; $\overline{\mathbf{C}} \in \mathbb{R}^{3n\times\tilde{n}}$ is the global stress-force matrix; $\mathbf{u} \in \mathbb{R}^d$ represents the corresponding structural displacement; $\mathbf{F}^{RI} \in \mathbb{R}^d$ is the uncertain externally applied force vector; $\mathbf{q}, \mathbf{e} \in \mathbb{R}^{\tilde{n}}$ are the independent internal force and its corresponding internal displacement respectively; $\boldsymbol{\sigma} \in \mathbb{R}^{3n}$ is the structural stress response; $\mathbf{S}_{\mathbf{e}}(\mathbf{E}^{RI}), \mathbf{S}_{\mathbf{v}}(\mathbf{E}^{RI}, \boldsymbol{\upsilon}^{RI}) \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$ denote the global flexibility matrices that are functions of the spatially dependent uncertain Young's modulus ($\mathbf{E}^{RI}$) and Poisson's ratio ($\boldsymbol{\upsilon}^{RI}$); $W_{\mu^I_{\mathbf{E}}}(\mathbf{x}, \omega_{\mathbf{E}})$, $W_{\mu^I_{\boldsymbol{\upsilon}}}(\mathbf{y}, \omega_{\boldsymbol{\upsilon}})$ and $W_{\mu^I_{\mathbf{F}}}(\mathbf{z}, \omega_{\mathbf{F}})$ denote the imprecise random fields for the considered uncertain parameters; $V_{\mathbf{E}}(\boldsymbol{\xi})$, $V_{\upsilon}(\boldsymbol{\tau})$, and $V_{\mathbf{F}}(\boldsymbol{\gamma})$ represent the associated interval fields for the considered uncertain parameters; '∨' is the logical 'or'.

However, direct solving Eq.(7) for $\mathbf{u}$ and $\boldsymbol{\sigma}$ is computationally intractable. For the purpose of valid uncertainty analysis, the following algorithm is proposed for estimating the PDFs and CDFs of the extreme structural responses.

Algorithm: The X-UISS method for structural uncertainty propagation analysis with multiple imprecise random and interval fields

Step 1. All considered imprecise random fields with interval mean are discretized to the *K*th order through the implementation of K-L expansion:

Step 2. All considered interval fields are discretized by implementing the spatial average method.

Step 3. Generate *b* (where, *b*>>1) realizations for the *K*th order approximated imprecise random field $\widehat{W}_{\mu^I}^{K}(\mathbf{x},\omega)$.

Step 4. Calculate the upper bound of the concerned structural response (e.g., the structural displacement at *s*th degree of freedom $\overline{u_s^r}$, where $1 \leq s \leq d$, or the *p*th stress component $\overline{\sigma_p^r}$, where $1 \leq p \leq 3n$).

Step 5. Calculate the lower bound of the concerned structural response (e.g., the structural displacement at *s*th degree of freedom $\overline{u_s^r}$, where $1 \leq s \leq d$, or the *p*th stress component $\overline{\sigma_p^r}$, where $1 \leq p \leq 3n$).

Step 6. Identification of the underpinned distribution for the extreme bounds of the concerned structural responses according to all collected samples.

Step 7. Estimations of PDFs and CDFs for the concerned structural responses through either parametric or non-parametric statistical inference.

5 Numerical Example

In order to demonstrate the effectiveness and efficiency of the proposed X-UISS approach, one numerical example is illustrated in the following section. In particular, all NLPs involved in this study are solved by a NLP solver named as CONOPT.

Here, a steel plate under uniform tension is considered. The general structural layout has been depicted in Figure 1(a), and the adopted finite element mesh is illustrated in Figure 1 (b). Without loss of generality, unit thickness is assumed for this particular example. The Young's modulus of the steel plate is considered as an imprecise Gaussian random field with an interval mean and exponential covariance function. In specific, $\mu_E^I = [193.580, 206.420]$, $\sigma_E = 30GPa$, and the correlation lengths in x- and y- directions are $L_x = 1$m and $L_y = 1$m respectively. On the other hand, due to the insufficiency of the information on the Poisson's ratio, it is modelled by the interval field with the following upper and lower bound functions respectively:

$$\overline{\nu}(x,y) = \frac{0.315\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} + 0.062\sqrt{x^2+y^2}\ , 0 < x \leq 1, 0 < y \leq 1 \tag{8}$$

$$\underline{\nu}(x,y) = \frac{0.285\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} + 0.062\sqrt{x^2+y^2}\ , 0 < x \leq 1, 0 < y \leq 1 \tag{9}$$

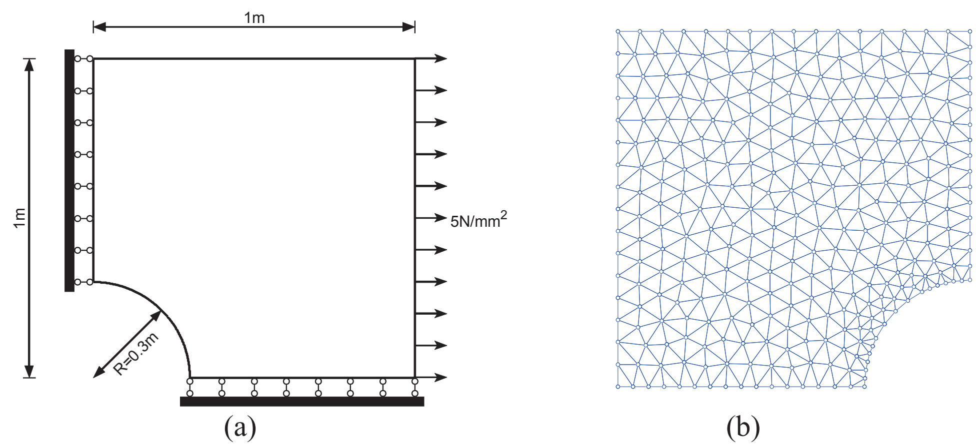


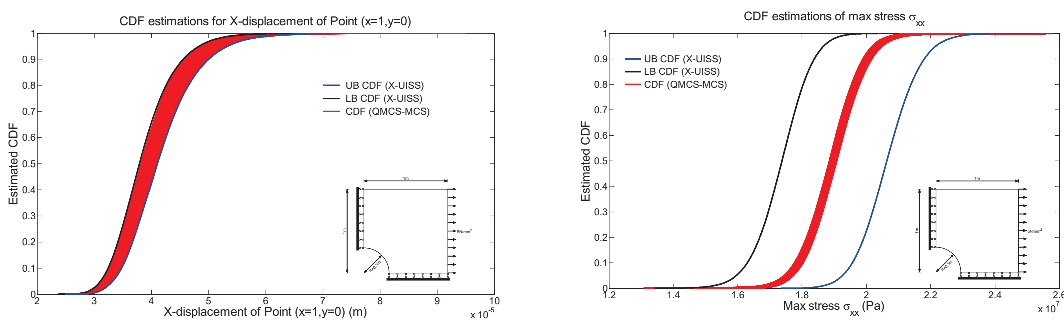
Figure 1. Steel plate under uniform tension (a) general structural layout (b) adopted mesh

In order to demonstrate the applicability of the proposed X-UISS approach for the structural analysis of the concerned steel plate involving both imprecise random and interval fields, both the X-UISS and the Quasi-Monte-Carlo Simulation combined with Monte-Carlo Simulation (i.e., QMCS-MCS) approaches are implemented to establish the CDFs of the concerned structural responses. In this example, the concerned structural responses are the horizontal displacement of the point at x=0, y=1, as well as the maximum normal stress in the horizontal direction that have been observed from the deterministic analysis. For all calculations involved in the proposed X-UISS approach, a sample size of 10,000 has been implemented. On the other hand, a total number of 2.5 million simulations have been adopted for the QMCS-MCS approach.

By utilizing the kernel density estimation, the CDFs of the bounds of the concerned structural responses can be efficiently established and presented in Figure 2. In addition to the results obtained from the proposed X-UISS approach, the estimations on the CDFs of the concerned structural responses by the QMCS-MCS are also being reported in Figure 2.

As clearly indicated in Figure 2, all possible CDFs reported by the QMCS-MCS approach have been enclosed by the CDFs established by the proposed X-UISS approach for both structural responses on displacement and stress with much less computational effort. That is, the proposed X-UISS has spent 17.7 hours for the establishment of the CDFs of the bounds of the concerned structural responses, whereas the QMCS-MCS approach has consumed in total of 986 hours to have all the red lines as indicated in Figure 2. Within the identical computational environment, the proposed method only consumed about 1.80% of the total computational effort required by the QMCS-MCS approach. Therefore, the proposed X-UISS approach surpasses the performance of the dually simulative computational scheme with extensive simulation cycles in both aspects of computational accuracy and efficiency.

In addition to the comparison on the computational results between two distinctive methods, one thing attracts the authors' attention is that a wider dispersion between the CDFs of the bounds of the maximum normal stress has been observed. That is, regarding the results calculated by the proposed X-UISS approach, the dispersion between the CDFs of the bounds of the concerned stress is much larger than the dispersion between the CDFs of the bounds of the concerned displacement. However, the trend of results reported by the QMCS-MCS approach does not share such similarity. The reason for such phenomenon is that the extreme bound of the concerned stress does not necessarily correspond to the extreme bound of the concerned displacement at each interval analysis. Within the scheme provided by the X-UISS approach, the extreme bounds of the displacement and stress are solved individually. However in the adopted QMCS-MCS approach, all the stresses are actually calculated basing on the displacement which was obtained in the interval analysis on the structural displacement. That is, all the stresses reported in Figure 2 (b) were actually calculated basing on the corresponding displacements determined in Figure 2 (a). Therefore, the variational dispersion exposed in Figure 2 within the framework of the X-UISS approach is actually highlighting additional advantage of the proposed method instead of introducing result overestimations.



(a) (b)

Figure 2. CDFs of the bounds of (a) the displacement (b) the maximum tensile stress

6 Conclusion

In this paper, the structural static analysis involving multiple imprecise random and interval fields has been investigated. Both stochastic and non-stochastic spatially dependent uncertain system parameters are incorporated simultaneously.

An effective computational method, namely X-UISS approach, is proposed by combing the robust sampling method with the mathematical programming approach. Through a thorough investigation on a simple plane stress problem, the applicability, accuracy and efficiency of the proposed method have been evidently illustrated.

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