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A NEW METHOD FOR SIMULATING NON-GAUSSIAN AND NONSTATIONARY STOCHASTIC PROCESSES

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A new method is developed for explicitly representing and synthesizing non-Gaussian and non-stationary stochastic processes that have been specified by their covariance function and marginal cumulative distribution function. The target process is firstly represented in the Karhunen-Loeve (K-L) series form, the random coefficients in the K-L series is subsequently decomposed using one-dimensional polynomial chaos (PC) expansion. In this way, the target process is represented in an explicit form, which is particularly well suited for stochastic finite element analysis of structures as well as for general purpose simulation of realizations of these processes. The key feature of the proposed method is that the covariance of the resulting process automatically matches the target covariance, and one only needs to iterate the marginal distribution to match the target one. Example is used to demonstrate the proposed method.

Keywords: Non-Gaussian, nonstationary, stochastic process simulation, Karhunen-Loeve expansion, Polynomial chaos.

1 Introduction

Problems involving random processes are commonly encountered in various fields of engineering. The modelling of parameters of input and/or system itself by means of stochastic processes increases significantly the size and complexity of the analysis. This is why the application of random field theory to engineering problems has gained considerable interest. For computational purposes, an analytical representation of stochastic processes in forms that are suitable for further mathematical manipulation is particularly preferred. Therefore, the efficient simulation of a general stochastic processes is of practical and theoretical importance.

In the past twenty years, the simulation of non-Gaussian and non-stationary stochastic processes has spawned the development of methods rooted in two different basic simulation algorithms: the spectral representation (SR) method (discretization in the frequency domain) and Karhunen-Loeve (K-L) expansion (discretization in the physical domain, e.g., time or space). Among them, Phoon et. al extended the K-L expansion to simulate sample functions of non-Gaussian process by iteratively updating the distribution of the underlying non-Gaussian K-L random variables [1]. This method has been further improved for simulating highly skewed non-Gaussian marginal distributions and thus offering a unified framework for the simulation of multi-dimensional homogeneous and non-homogeneous random fields [2]. Other techniques have also been developed for simulation of non-Gaussian processes. One important group of such techniques are those utilizing polynomial chaos (PC) decomposition such as those by Sakamoto and Ghanem [3]. The method represents the process as a multidimensional Hermite polynomial in a set of normalized Gaussian variables.

Based on the K-L and PC expansion, this paper presents a new method for explicitly simulating non-Gaussian and non-stationary stochastic process that has been specified by its covariance function and marginal non-Gaussian CDF. The basic idea is to firstly represent the target process in the K-L series form, and then expand the random coefficients in K-L series using one-dimensional PC expansion. In this way, the target process is represented in an explicit form, which is particularly well suited for SFE analysis of structures in the calculation of response variability as well as the simulation of realizations of stochastic processes.

2 The new simulation method

2.1 *Polynomials chaos representation of random variables*

An arbitrary second-order random variable $u(\theta)$ can be represented in the form:

$$u(\theta) = a_0\Gamma_0 + \sum_{i_1=1} a_{i_1}\Gamma_1\left(\gamma_{i_1}(\theta)\right) + \sum_{i_1=1}\sum_{i_2=1} a_{i_1 i_2}\Gamma_2\left(\gamma_{i_1}(\theta),\gamma_{i_2}(\theta)\right) + \cdots, \tag{1}$$

where $\Gamma_j(\gamma_{i_1}(\theta),\cdots,\gamma_{i_j}(\theta))$ denotes the Hermite chaos of order j in the variables $(\gamma_{i_1},\cdots,\gamma_{i_j})$ and Γ_j are Hermite polynomials in terms of the standard Gaussian variables γ with zero mean and unit variance. The multidimensional Γ_j are generated by the formula of Rodriguez:

$$\Gamma_j(\gamma) = (-1)^j e^{\gamma^T\gamma/2}\frac{\partial^j}{\partial\gamma_{i_1}\cdots\partial\gamma_{i_j}} e^{-\gamma^T\gamma/2} \tag{2}$$

Correspondingly, the one-dimensional Hermite polynomials are generated as

$$\Gamma_j(\gamma) = (-1)^j\frac{\varphi^{(j)}(\gamma)}{\varphi(\gamma)} \tag{3}$$

where $\varphi^{(j)}(\gamma)$ is the j-th derivative of the normal density $\varphi(\gamma) = 1/\sqrt{2\pi}\, e^{-\gamma^2/2}$. This is simply the single-variable version. From Eq. (3), one can readily find $\{\Gamma_j\} = \left\{1,\ \gamma,\ \gamma^2-1,\ \gamma^3-3\gamma\cdots\right\}$. The orthogonal polynomials Γ_i and standard Gaussian variable γ satisfy

$$\begin{aligned} &\Gamma_0 = 1, \quad \left\langle\Gamma_i\Gamma_j\right\rangle = \left\langle\Gamma_i^2\right\rangle\delta_{ij} \quad \forall i,j \\ &\left\langle\gamma^0\right\rangle = 1, \quad \left\langle\gamma^k\right\rangle = 0 \ \ \forall k \text{ odd}, \quad \text{and } \left\langle\gamma^k\right\rangle = (k-1)\left\langle\gamma^{k-2}\right\rangle \end{aligned} \tag{4}$$

where $\langle\cdot\rangle$ denotes the inner product operator. For example, when using one-dimensional PC expansion, the random variable $u(\theta)$ can be represented in a mean-square convergent series as

$$u(\theta) = \sum_{i=0} a_i\Gamma_j(\gamma) = a_0 + a_1\gamma + a_2(\gamma^2-1) + a_3(\gamma^3-3\gamma) + \cdots \tag{5}$$

where the deterministic coefficients a_i in the expansion can be determined by virtue of the orthogonality of the approximating polynomials with respect to the Gaussian measure as

$$a_i = \frac{\left\langle u(\theta)\Gamma_i(\gamma)\right\rangle}{\left\langle\Gamma_i^2(\gamma)\right\rangle} \tag{6}$$

Generally, the less number of the standard Gaussian variables of the Hermite polynomials, the more efficient the PC expansion approximates the random variable. Therefore, it will be very efficient to use the one-dimensional PC expansion to approximate non-Gaussian random variables, although the approximation accuracy may not be guaranteed in some cases. However, the one-dimensional PC expansion can not accurately represent the 'strongly non-Gaussian' random variable although the order of the Hermite polynomial is adopted very high.

2.2 *Proposed method with KL and PC expansions*

The basic idea of the proposed method is to firstly represent the target process in the K-L series form, and then expand the random coefficients in the K-L series with one-dimensional PC expansion. Consider a stochastic process $w(x,\theta)$ specified by its covariance function $C(x_1,x_2)$ and marginal CDF $F(x,y)$. The method first decomposes the process using K-L expansion as

$$\tilde{\omega}(x,\theta)=\bar{\omega}(x)+\sum_{i=1}^{M}\sqrt{\lambda_i}\xi_i(\theta)f_i(x) \tag{7}$$

where $\bar{\omega}(x)$ is the mean function of the process, and M is the number of the K-L series. For each random variable $\xi_i(\theta)$ in Eq. (7), the following one-dimensional PC expansion

$$\xi(\theta)=\sum_{j=1}^{P}a_j\Gamma_j(\gamma) \tag{8}$$

is further used to simulate $\xi_i(\theta)$, resulting in the equation

$$\tilde{\omega}(x,\theta)=\bar{\omega}(x)+\sum_{i=1}^{M}\sum_{j=1}^{P}a_{ij}\Gamma_j(\gamma_i)\sqrt{\lambda_i}f_i(x) \tag{9}$$

where P is the order of the one-dimensional Hermite polynomials as given in Eq. (3). Obviously, the summation for i represents the K-L expansion of $w(x,\theta)$ and summation for j represents the one-dimensional PC expansion of variable $\xi_i(\theta)$. With these two round of approximation, a new explicit representation of the stochastic process $w(x,\theta)$ is derived as shown in Eq. (9).

Since the accuracy for simulating non-Gaussian processes when using Eq. (9) can not be guaranteed, Eq. (9) is reconsidered. The basic idea is to consider the product of $\sqrt{\lambda_i}$, $f_i(x)$ and $\Gamma_j(\gamma_i)$ as deterministic basis functions and $\Gamma_j(\gamma_i)$ are the one-dimensional Hermite polynomials, and the only random coefficients need to be determined are a_{ij}. Obviously, the process $w(x,\theta)$ can be explicitly synthesised as long as a_{ij} are known. With the orthogonality of the Hermite polynomials as given in Eq. (4), the expectation of $\xi_i(\theta)$ is derived as

$$\left\langle\xi_i(\theta)\right\rangle=\left\langle\sum_{j=1}^{P}a_{ij}\Gamma_j(\gamma_i)\right\rangle=\sum_{j=1}^{P}a_{ij}\left\langle\Gamma_j(\gamma_i)\right\rangle=0 \tag{10}$$

and the covariance of the random variables $\xi_i(\theta)$ and $\xi_j(\theta)$ is computed as

$$\begin{aligned}\left\langle\xi_i(\theta)\xi_j(\theta)\right\rangle&=\left\langle\left\{\sum_{j=1}^{P}a_{ij}\Gamma_j(\gamma_i)\right\}\left\{\sum_{j=1}^{P}a_{kj}\Gamma_j(\gamma_k)\right\}\right\rangle\\&=\sum_{l=1}^{P}\sum_{j=1}^{P}a_{il}a_{kj}\left\langle\Gamma_l(\gamma_i)\Gamma_j(\gamma_k)\right\rangle=\sum_{l=1}^{P}\sum_{j=1}^{P}a_{il}a_{kj}\delta_{lj}\delta_{ik}\left\langle\Gamma_l\Gamma_j\right\rangle=\sum_{j=1}^{P}a_{ij}^2\left\langle\Gamma_j^2\right\rangle\end{aligned} \tag{11}$$

Therefore, the random coefficients a_{ij} have to be satisfied the following condition so that the covariance $\left\langle\xi_i(\theta)\xi_j(\theta)\right\rangle$ has the unit variance

$$\sum_{j=1}^{P}a_{ij}^2\left\langle\Gamma_j^2\right\rangle=1 \tag{12}$$

With the condition in Eq. (12) holds, the covariance function of the simulated process $\tilde{\omega}(x,\theta)$ can be readily derived as

$$\begin{aligned} C_{\tilde{w}}(x_1,x_2) &= \sum_{i=1}^{M}\left[\sum_{j=1}^{P} a_{ij}\Gamma_j(\gamma_i)\right]^2 \lambda_i f_i(x_1) f_i(x_2) \\ &= \sum_{i=1}^{M}\sum_{j=1}^{P} a_{ij}^2 \left\langle \Gamma_j^2 \right\rangle \lambda_i f_i(x_1) f_i(x_2) = \sum_{i=1}^{M} \lambda_i f_i(x_1) f_i(x_2) \approx C(x_1,x_2) \end{aligned} \tag{13}$$

Eq. (13) demonstrates the advantage of the proposed method, i.e., when using Eq. (9) to explicitly represent the target process, the covariance of the simulated process automatically matches that of the target one with the increase of K-L series terms *M*. From Eq. (9), we have

$$\int_D \left[\omega(x,\theta)-\bar{\omega}(x)\right] f_k(x)\,dx = \sum_{i=1}^{M}\sum_{j=1}^{P} a_{ij}\Gamma_j(\gamma_i)\sqrt{\lambda_i}\int_D f_i(x) f_k(x)\,dx \tag{14}$$

Since the eigenfunctions are orthogonal, Eq. (14) can be simplified as

$$\frac{1}{\sqrt{\lambda_i}}\int_D \left[\omega(x,\theta)-\bar{\omega}(x)\right] f_i(x)\,dx = \sum_{j=1}^{P} a_{ij}\Gamma_j(\gamma_i) \tag{15}$$

By multiplying the one-dimensional Hermite polynomials $\Gamma_m(\gamma_n)$ and taking the inner product on both sides of Eq. (15), we have

$$\frac{1}{\sqrt{\lambda_i}}\left\langle \Gamma_m(\gamma_n)\int_D \left[\omega(x,\theta)-\bar{\omega}(x)\right] f_i(x)\,dx \right\rangle = \sum_{j=1}^{P} a_{ij}\left\langle \Gamma_j(\gamma_i)\Gamma_m(\gamma_n)\right\rangle \tag{16}$$

Due to the orthogonal property of the Hermite polynomials, coefficients a_{ij} are then derived as

$$a_{ij} = \frac{1}{\left\langle \Gamma_j^2 \right\rangle \sqrt{\lambda_i}}\int_D \left\langle \left[\omega(x,\theta)-\bar{\omega}(x)\right]\Gamma_j(\gamma_i)\right\rangle f_i(x)\,dx \tag{17}$$

Thus, by analytically deriving the formula of the random coefficients a_{ij}, the original stochastic process can be explicitly represented using Eq. (9). In this way, the aforehand-mentioned issue about accuracy for simulating highly non-Gaussian variable existed in the one-dimensional PC expansion can be avoided. One can simulate target non-Gaussian stochastic process with high accuracy through developing iteration algorithm to obtain optimal coefficients a_{ij}.

2.3 *Simulation algorithm for non-Gaussian processes*

The proposed algorithm for simulating non-Gaussian processes is described as follows:

Step 1: Decompose the covariance of the target process $w(x,\theta)$ into its eigenpairs $\sqrt{\lambda_i}$ and $f_i(x)$. Generate *N* set of Gaussian random variables, a total of *M* variables for each set, and compute corresponding *N* set of one-dimensional Hermite polynomials of order *P* using Eq. (3).

Step 2: Generate *N* sample functions of the non-Gaussian process using Eq. (9) as

$$w_m^{(k)}(x,\theta) = \bar{\omega}(x) + \sum_{i=1}^{M}\sum_{j=1}^{P} a_{ij}^{(k)}\Gamma_j(\gamma_i)\sqrt{\lambda_i} f_i(x) \quad m = 1\cdots N \tag{18}$$

where *k* is the iteration number, and *m* is the sample number. The initial value of random coefficients a_{ij} is set based on Eq. (12).

Step 3: Compute the simulated marginal CDF as

$$G^{(k)}(y/x) = \frac{1}{N}\sum_{m=1}^{N} I[\hat{w}^{(k)}(x,\theta) \le y] \tag{19}$$

It should be noted that, one does not need to compute the simulated covariance since it automatically matches that of the target process according to Eq. (13).

Step 4: Since the simulated marginal CDF $G^{(k)}(y/x)$ generally does not agree with the target one, the coefficients a_{ij} need to be modified so that simulated process follows the target marginal CDF. This requires to firstly transform each sample function as

$$\phi_m^{(k)}(x,\theta) = F^{-1}\left\{G^{(k)}\left[\omega_m^{(k)}(x,\theta)\right]\right\} \quad m = 1\cdots N \tag{20}$$

and then update the coefficients a_{ij}

$$a_{ij}^{(k+1)} = \frac{1}{\left\langle \Gamma_j^2 \right\rangle \sqrt{\lambda_i}} \int_D \left\langle \left[\phi^{(k)}(x,\theta) - \bar{\phi}^{(k)}(x)\right] \Gamma_j(\gamma_i) \right\rangle f_i(x)\, dx \tag{21}$$

Step 5: Steps 2 through 4 are repeated until the maximum value of two successive iterations for the coefficients a_{ij} is limited within the tolerance $\max\left\{\left|a_{ij}^{(k+1)} - a_{ij}^{(k)}\right|\right\} < \varepsilon$.

3. Simulation of non-stationary and strongly non-Gaussian process

Consider a zero-mean process $w(x,\theta)$ with covariance function given by

$$C(x_1,x_2) = \min(x_1,x_2), \quad (x_1,x_2) \in [0,1]\times[0,1] \tag{22}$$

Since the covariance is of the type of Wiener process, this example can be used to investigate the capacity of the proposed method for simulating non-stationary stochastic processes. According to [4], the eigenvalues and eigenfunctions can be solved analytically. Therefore, the covariance of the simulated process can be computed based on Eq. (13) as

$$C_M(x_1,x_2) = \sum_{n=1}^{M} \frac{8}{(2n-1)^2\pi^2} \sin\left((n-\frac{1}{2})\pi x_1\right) \sin\left((n-\frac{1}{2})\pi x_2\right) \tag{23}$$

According to [4], the simulated covariance in Eq. (23) approaches to the target covariance with the number of K-L series terms *M* large enough. In this example, *M* is adopted as 10. Thus, the simulated covariance using the proposed method automatically match the exact covariance as given in Eq. (22). The marginal non-Gaussian CDF is shifted log-normal distributed, with the CDF given by

$$F(y;u,\sigma,\delta) = \Phi\left[\frac{\ln(y+\delta(x)) - \mu(x)}{\sigma}\right] \tag{24}$$

where the scaling parameter μ and the position parameter δ are functions of *x*. In this example, all distribution parameters are chosen the same as in [1], i.e., the shape parameter σ is chosen to be one, the scaling parameter and the position parameter are respectively chosen as $\mu(x) = 0.5\ln x - 0.7707$ and $\delta(x) = 0.582x$, so that the target mean of the distribution is zero. In this example, the order of the one-dimensional Hermite polynomials *P* is still adopted as 10. Fig. 1 shows the exact and the simulated marginal CDF of the target process. It is seen that, besides the small difference in the tail distribution, the simulated marginal CDF is in good accordance with the exact one, demonstrating the high accuracy of the proposed method for simulating non-Gaussian and nonstationary stochastic processes.

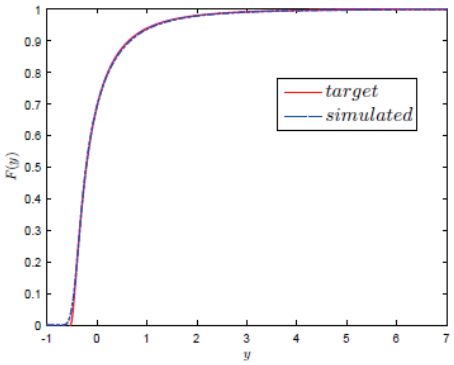

Fig.1 Exact and simulated marginal CDF of Winner process

4. Conclusions

A novel methodology is presented for explicitly simulating non-Gaussian and non-stationary stochastic processes. The method analytically represents the target process through combination of the K-L and PC expansion. By virtue of the orthogonal property of the Hermite polynomials with respect to the Gaussian measure, the covariance of the resulting process automatically matches the target covariance, and one only needs to iterate the marginal CDF to match the target one. Thus, the new method offers an explicit and unified framework for simulation of non-Gaussian processes with arbitrary covariance function. Numerical example is finally used to demonstrate the effectiveness and range of applicability of the proposed method.

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