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AN INTEGRATED THIRD-MOMENT METHOD FOR STRUCTURAL RELIABILITY WITH CORRELATED INPUT VARIABLES

Chao-Huang Cai[1], Zhao-Hui Lu[1,3*] and Yan-Gang Zhao[2,3]

[1] *School of Civil Engineering, Central South Univ., 22 Shaoshannan Rd., Changsha 410075, China. E-mail: caichaohuang@csu.edu.cn*

[2]*Dept. of Architecture and Building Engineering, Kanagawa Univ., 3-27-1 Rokkakubashi, Kanagawa-ku, Yokouhama 221-8686, Japan. E-mail: zhao@kanagawa-u.ac.jp*

[3]*Key Laboratory of Urban Security and Disaster Engineering of Ministry of Education, Beijing University of Technology, Beijing 100124, China*

(Corresponding author). E-mail: luzhaohui@csu.edu.cn

One of routes to perform structural reliability analysis is the high-order moment methods. This study proposes a third-order moment reliability method for performance function involving correlated random variables with distributions unknown. The main procedure of the proposed method includes three steps. First, the correlated random variables are transformed into independent standard normal random variables by the third-moment transformation. Second, with the aid of the bivariate dimensional reduction method, the performance function is approximated by a summation of one-dimensional functions and two-dimensional functions. And then, the moments of one-dimensional functions and two-dimensional functions without cross terms are determined by one-dimensional point estimate method, while the moments of two-dimensional function with cross terms are obtained using sparse grid stochastic collocation method. Third, the reliability index of the performance function involving correlated random variables is determined by a third-moment reliability index. Several numerical examples are presented to illustrate the efficiency, accuracy, and applicability of the proposed method.

Keywords: structural reliability assessment, correlated random variables, third-order reliability index, third-moment transformation.

1 Introduction

A fundamental problem in structural reliability analysis is the computation of the probability of failure, which involves multi-fold probability integral. Difficult in computing this integral has led to the development of various approximation methods. One of routes to perform structural reliability analysis is the high-order moment methods (Zhao and Ono 2001), in which evaluation of statistical moments of performance function is one of the main topics. Recently, the point estimate methods (Zhao and Ono 2000) based on normal transformations, e.g., Rosenblatt transformation (Rackwitz and Fiessler 1978) and Nataf transformation (Der Kiureghian and Liu 1986), and dimensional reduction methods, e.g., univariate- and bivariate-dimension reduction method (Zhao and Ono 2000; Xu and Rahman 2004), have been proposed to evaluate the statistical moments. In these methods, the joint probability density function (PDF) or marginal PDFs and correlation matrix of correlated random variables are assumed to be known. However,

the joint PDF and marginal PDFs of random variables encountered in engineering practice are often unknown. Moreover, studies show that, from the perspective of balancing accuracy and efficiency, evaluating the statistical moments of performance function still remains a challenge.
In the present paper proposes a third-order moment reliability method, in which an efficient algorithm method for evaluating statistical moments of performance function is developed. The main procedure of the proposed method includes three steps. First, based on the third-moment transformation, the correlated random variables are firstly transformed into independent standard normal random variables, in which only the first three moments and correlation matrix are required. Second, with the aid of the bivariate dimensional reduction method, the performance function is approximated by a summation of one-dimensional functions and two-dimensional functions. And then, the moments of one-dimensional functions and two-dimensional functions excluding cross terms are determined by one-dimensional point estimate method, while the moments of two-dimensional function including cross terms are obtained using sparse grid stochastic collocation method (SGSCM). Third, the reliability index of performance function involving correlated random variables is determined by a third-moment reliability index. The results demonstrate that the proposed method achieves a good balance between accuracy and efficiency, and provides a useful tool for structural reliability analysis involving correlated random variables, especially when the distributions of basic random variables are unknown.

2 The Third-Moment Transformation for Correlated Random Variables

Without loss of generality, an arbitrary random variable X_i can be standardized as follows:

$$X_{is} = (X_i - \mu_{X_i}) / \sigma_{X_i} \tag{1}$$

where X_i is the ith random variable of correlated random vector $\mathbf{X}$; X_{is} is the standardized random variables of X_i; and μ_{X_i} and σ_{X_i} are the mean and standard deviation of X_i, respectively.
According to the third-moment transformation (Lu et al. 2017), the standardized variable X_{is} can be approximated by a second-order polynomial normal function, which is formulated as:

$$X_{is} = S_z(Z_i) = a_i + b_i Z_i + c_i Z_i^2 \tag{2}$$

where Z_i is the ith random variable of correlated standard normal vector $\mathbf{Z}$; $S_z(Z_i)$ is the second-order polynomial of Z_i; and a_i, b_i, and c_i are the polynomial coefficients.
Making the first three moments of $S_z(Z_i)$ equal to those of X_{is}, the polynomial coefficients a_i, b_i, and c_i can be determined as (Zhao and Ono 2000):

$$c_i = -a_i = \mathrm{Sign}(\alpha_{3X_i})\sqrt{2}\cos\left[\frac{|\theta_i|+\pi}{3}\right];\ b_i = \sqrt{1-2c_i^2}\ ;\ \theta_i = \tan^{-1}\left(\frac{\sqrt{8-\alpha_{3X_i}^2}}{\alpha_{3X_i}}\right) \tag{3}$$

where α_{3X_i} is the skewness (third-order central moment) of X_i.

Assume that the correlation coefficient between X_i and X_j is ρ_{ij}, and the correlation coefficient between Z_i and Z_j is ρ_{0ij}. According to the definition of correlation coefficient, leads to

$$\rho_{ij} = \sqrt{(1-2c_i^2)(1-2c_j^2)}\rho_{0ij} + 2c_i c_j \rho_{0ij}^2 \tag{4}$$

The valid solution of ρ_{0ij} should be restricted by the condition, $-1 \le \rho_{0ij} \le 1$ and $\rho_0 \cdot \rho_{0ij} \ge 0$, to satisfy the definition of the correlation coefficient.

From the preceding discussion, any two arbitrary correlated random variables with known first three moments and correlation coefficient can be approximated by two correlated standard normal variables. The polynomial coefficients of each variable can be obtained by Eq. (3), and for any two correlated variables, the corresponding equivalent correlation coefficients of standard normal variables, ρ_{0ij}, can be determined from solving Eq. (4). Then, the equivalent correlation matrix of standard normal variables, $\mathbf{C_Z}$, can then be summarized. If the correlation matrix $\mathbf{C_Z}$ is a positive-definite matrix, using Cholesky decomposition, it can be rewritten as $\mathbf{C_Z} = \mathbf{L}_0\mathbf{L}_0^T$, in which $\mathbf{L}_0$ is a lower triangular matrix and $\mathbf{L}_0^T$ is the transpose matrix of $\mathbf{L}_0$. Then the relationship between the correlated standard normal random vector $\mathbf{Z}$ and the independent standard normal random vector $\mathbf{U}$ can be expressed as:

$$\mathbf{Z} = \mathbf{L}_0\mathbf{U} \tag{5}$$

According to Eqs. (1), (2), and (5), the relation between $\mathbf{X}$ and $\mathbf{U}$ can be expressed as:

$$X_i = \mu_{X_i} + \sigma_{X_i}\left[a_i + b_i\sum_{k=1}^{i} l_{ik}U_k + c_i\left(\sum_{k=1}^{i} l_{ik}U_k\right)^2\right], \quad (i = 1,\ 2,\ \cdots,\ n) \tag{6}$$

where l_{ik} is the ith row kth column element of $\mathbf{L}_0$.

3 Evaluation of the First Three Moments of Performance Function Involving Correlated Random Variables

Substituting Eq. (6) into performance function $G(\mathbf{X})$, in which $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$ is the basic random variables with the correlation matrix $\mathbf{C_X}$, it can be formulated as:

$$G(\mathbf{X}) = G(X_1, X_2, \cdots, X_n) = g(U_1, U_2, \cdots, U_n) = g(\mathbf{U}) \tag{7}$$

where $g(\mathbf{U})$ is a function with inclusion of independent standard normal random variables only. Based on the bivariate dimension-reduction method (Xu and Rahman 2004), the function $g(\mathbf{U})$ can be approximated as:

$$g(\mathbf{U}) = g_2 - (n-2)g_1 + \frac{(n-1)(n-2)}{2}g_0 \tag{8}$$

where

$$g_0 = g(0,0,\cdots,0)\,;\ g_1 = \sum_{i=1}^{n} g(0,0,\cdots,0,U_i,0,\cdots,0) = \sum_{i=1}^{n} g_{1,i}(U_i) \tag{9a}$$

$$g_2 = \sum_{i<j} g(0,\cdots,0,U_i,0,\cdots,0,U_j,0,\cdots,0) = \sum_{i<j} g_{2,ij}(U_i,U_j) \tag{9b}$$

According to the definition of statistical moments and Eq. (8), the kth row moments of the performance function $G(\mathbf{X})$ can be approximated as:

$$\mu_{kG} = E\left\{[G(\mathbf{X})]^k\right\} = E\left\{[g(\mathbf{U})]^k\right\} \approx \sum_{i<j}\mu_{k-g_{2,ij}} - (n-2)\sum_{i=1}^{n}\mu_{k-g_{1,i}} + \frac{(n-1)(n-2)}{2}g_0^k \tag{10}$$

where E denotes expectation;
Based on the point estimate method in independent normal space (Zhao and Ono 2000), leads to:

$$\mu_{k-g_{1,i}} = \sum_{r=1}^{m} P_r \left[g_{1,i}(u_r) \right]^k \tag{11}$$

where u_r and P_r are the estimating points and corresponding weights, respectively.
According to the criterion for delineating the existence of cross terms in two-dimensional functions (Fan et al. 2016), the two-dimensional functions $g_{2,\,ij}(U_i, U_j)$ can be decomposed as functions that include or exclude cross terms of U_i and U_j. In this study, for the two-dimensional functions without cross terms, their statistical moments are directly obtained from the moments of two one-dimensional functions. While for the two-dimensional functions with cross terms, their statistical moments are obtained by two-dimensional SGSCM. For a two-dimensional function $g_{2,\,ij}(U_i, U_j)$ without the cross terms of U_i and U_j, using the univariate dimension-reduction method, it can be expressed as $g_{2,ij}(U_i,U_j) = g_{1,i}(U_i) + g_{1,j}(U_j) - g_0$. Then, the first three moments of $g_{2,\,ij}(U_i, U_j)$ can be directly obtained using the first three moments of $g_{1,i}(U_i)$ and $g_{2,j}(U_j)$ as follow:

$$\mu_{1-g_{2,ij}} = \mu_{1-g_{1,i}} + \mu_{1-g_{1,j}} - g_0 \,;\; \mu_{2-g_{2,ij}} = M_{2ij} + \mu_{1-g_{2,ij}}^2 \,;\; \mu_{3-g_{2,ij}} = M_{3ij} + 3\mu_{2-g_{2,ij}} \cdot \mu_{1-g_{2,ij}} - 2\mu_{1-g_{2,ij}}^3 \tag{12}$$

where $M_{2ij} = \sum_k (\mu_{2-g_{1,k}} - \mu_{1-g_{1,k}}^2)$; $M_{3ij} = \sum_k (\mu_{3-g_{1,k}} - 3\mu_{2-g_{1,k}} \cdot \mu_{1-g_{1,k}} + 2\mu_{1-g_{1,k}}^3)$, in which $\mu_{1-g_{1,k}}$, $\mu_{2-g_{1,k}}$, and $\mu_{3-g_{1,k}}$ are the first three moments of $g_{1,k}(U_k)$ ($k = i, j$).
With the aid of the Smolyak-type algorithm (Smolyak 1963), the kth row moment of a two-dimensional function $g_{2,ij}(U_i, U_j)$ with cross term of U_i and U_j can be derived as (He et al. 2014):

$$\mu_{k-g_{2,ij}} = \sum_{\boldsymbol{i} \in H(q,2)} (-1)^{q+2-|\boldsymbol{i}|} \begin{pmatrix} 1 \\ q+2-|\boldsymbol{i}| \end{pmatrix} \sum_{r_1=1}^{2i_1-1} \sum_{r_2=1}^{2i_2-1} P_{r_1} \cdot P_{r_2} \left[g_{2,ij}(u_{r_1}, u_{r_2}) \right]^k \tag{13}$$

where $u_{r_h} = \sqrt{2} x_{r_h}$; $P_{r_h} = \omega_{r_h} \big/ \sqrt{\pi}$ (h = 1, 2), in which x_{r_h} and ω_{r_h} are respectively the abscissas and weights of $(2i_h - 1)$ order Gauss-Hermite integration with the weight function $\exp(-x^2)$; $\boldsymbol{i} = (i_1, i_2) \in \mathbb{N}_+^2$, $|\boldsymbol{i}| = i_1 + i_2$; q is a nonnegative integer; and $H(q, 2)$ is defined as:

$$H(q,2) = \left\{ \boldsymbol{i} \in \mathbb{N}_+^2, \boldsymbol{i} \geq (1,1) : q+1 \leq i_1 + i_2 \leq q+2 \right\} \tag{14}$$

Substituting all of the first three raw moments of one- and two-dimensional functions into Eq. (10), the first three raw moments of performance function involving correlated random variables can be determined. Then the first three central moments can be determined as following:

$$\mu_G = \mu_{1G} \,;\; \sigma_G = \sqrt{\mu_{2G} - \mu_{1G}^2} \,;\; \alpha_{3G} = \left(\mu_{3G} - 3\mu_{2G}\mu_{1G} + 2\mu_{1G}^3 \right) \big/ \sigma_G^3 \tag{15}$$

4 Third-order moment reliability index

According to the three-parameter lognormal distribution (Tichy 1994), the third-order reliability index is obtained as (Zhao and Ono 2001):

$$\beta_{TM} = -\frac{\mathrm{Sign}(\alpha_{3G})}{\sqrt{\ln(A)}} \ln \left[\sqrt{A} \left(1 + \frac{\beta_{SM}}{u_b} \right) \right] \tag{16a}$$

$$\beta_{SM}=\frac{\mu_G}{\sigma_G};\ A=1+\frac{1}{u_b^2};\ u_b=\sqrt[3]{a_0+b_0}+\sqrt[3]{a_0-b_0}-\frac{1}{\alpha_{3G}} \tag{16b}$$

$$a_0=-\frac{1}{\alpha_{3G}}\left(\frac{1}{2}+\frac{1}{\alpha_{3G}^2}\right);\ b_0=\frac{1}{2\alpha_{3G}^2}\sqrt{\alpha_{3G}^2+4} \tag{16c}$$

5 Numerical Examples and Investigations

In order to investigate the efficiency and accuracy of the proposed methods for structural reliability analysis involving correlated random variables, two numerical examples are investigated in this section.

5.1 *Example 1: Investigation of the influence of transformation order*

The first example considers the following performance function:

$$G(\mathbf{X})=18-3X_1-2X_2 \tag{17}$$

where the joint PDF of X_1 and X_2 is $f_{\mathbf{X}}(\mathbf{x})=(x_1+x_2+x_1x_2)\exp(-x_1-x_2-x_1x_2)$, for $x_1 \geq 0$, and $x_2 \geq 0$.
For this example, because the joint PDF and performance function are available, the probability of failure, P_f, can be obtained directly by numerical integral as P_f = 2.9449×10^{-3}, and the corresponding reliability index is 2.75. With the aid of the joint PDF of X_1 and X_2, the first three moments and correlation coefficient of X_1 and X_2 can be determined as $\mu_{X1}=\mu_{X2}=1$, $\sigma_{X1}=\sigma_{X2}=1$, α_{3X1} = α_{3X2} = 2, and ρ_{12} = -0.403653. Based on the third-moment transformation, if the transformation order $X_1 \rightarrow X_2$ used, the performance function G(X) can be approximated as:

$$G(\mathbf{X})=g_1(\mathbf{U})=14.83-1.26U_1-1.53U_1^2+0.72U_1U_2-1.10U_2-0.30U_2^2 \tag{18}$$

Using the proposed method, the first three central moments corresponding to Eq. (18) are obtained as μ_G = 13.0, σ_G = 2.8559, and α_{3G} = -1.9379. According to Eq. (16a), the third-order reliability index obtained as β_{TM} = 2.668. It can be observed that the reliability index is good agreement with the exact result (obtained by numerical integral). Similarly, if the transformation order $X_2 \rightarrow X_1$ used, the third-moment reliability index is also 2.668. It can be observed that the results of the proposed method are the same even when the transformation order is different.

5.2 *Example 2: A 61-bar truss structure with an implicit performance function*

The second example considers a 61-bar truss structure as shown in Fig. 1. The cross section areas of all members are deterministic and identical with 1.229×10^{-3} m^2. The Young's modulus E and concentrated loads F_i (i = 1, 2, …, 13) are random variables with the first three moments and correlation matrix known. And their first three moments are respectively μ_E = 210 GPa, σ_E = 21GPa, α_{3E} = 0.301, μ_F = 12 kN, σ_F = 3.0 kN, and α_{3F} = 0.766. E is independent of all F_i and F_i are correlated with correlation coefficient $\rho_{\mathbf{F}}$ = 0.1. The reliability problem in this example is to determine the probability of failure that the maximum value of the vertical displacement of nodes may exceed in magnitude specified limit, $u_{\lim}$, thus the performance function is defined as:

$$G(\mathbf{X})=u_{\lim}-\max\{|u_{yi}|\} \tag{26}$$

where $u_{\lim} = L/500 = 36$mm; and u_{yi} is the vertical displacement of ith node, which is determined by finite element analysis.
Because the joint PDF and marginal PDFs of basic random variables are unknown, the reliability analysis based on Rosenblatt transformation and Nataf transformation cannot be applied. However, using the proposed method, the third-order reliability can be easily obtained as 3.567.

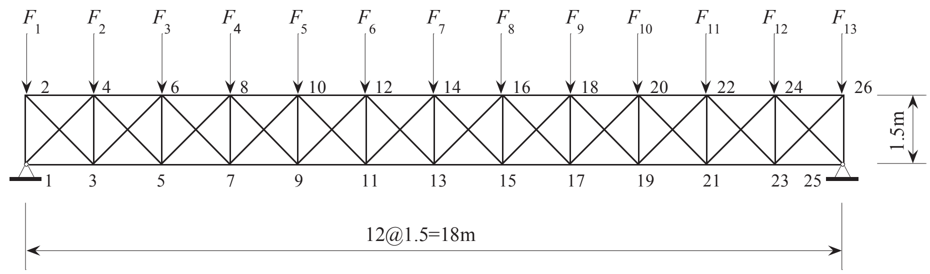

Figure 1. A 61-bar truss structure for Example 2

6 Conclusions

The present paper proposed a third-order moment method for structural reliability involving correlated random variables. From the investigation of this paper, the following conclusions can be draw: The proposed third-order moment method, being very simple, has no shortcoming of varying with the transformation order of random variables and thus is convenient to be applied to structural reliability analysis. Because the proposed method is based on the first three moments and correlation matrix of basic random variables, it can be applied for structural reliability assessment even when the joint PDF and marginal PDFs of the basic random variables are unavailable.

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