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# AN ANALYTICAL CONDITIONAL RANDOM FIELD SAMPLING APPROACH FOR SLOPE RELIABILITY ANALYSIS

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Abstract: To properly characterize the spatial variation, this paper proposes an analytical approach for sampling conditional random field of soil undrained shear strength. With the proposed approach, the posterior statistics of spatially varying undrained shear strength conditioned on the known values at measurement locations can be solved analytically. The conditional random field model of undrained shear strength is constructed using the field vane shear test data at a site of the west side highway in New York. The probability of slope failure is estimated by subset simulation. A clay slope excavated at this site is investigated as an example to illustrate the proposed approach.

*Key words: slope reliability, undrained shear strength, spatial variability, conditional random field*

## 1 Introduction

The inherent spatial variability of soil properties has been well accounted for by random field theory in the slope stability analyses (e.g., El-Ramly et al., 2002; Griffiths and Fenton, 2004). The statistical information [e.g., means, coefficients of variation (COVs), distributions and scales of fluctuation] of the soil properties are usually inferred from geotechnical testing data (e.g., Fenton, 1999), however, the random fields do not necessarily incorporate the specific “known” properties (and thus considered “certain”), albeit limited, as measured from the project site. To make best use of available data to characterize the spatial variability of soil properties, it is of significance to construct the random fields conditioned on the site-specific data.

The conditional random fields can take account of the statistical information of soil properties as well as the soil properties known at measurement locations to model the spatial variability. Various numerical approaches have been developed to sample the conditional random fields (e.g., Lloret-cabot et al., 2012; Li et al., 2016; Liu et al., 2017; Gong et al., 2018), however, the computational efficiency is low for high-dimensional problems. This study proposes an analytical approach for sampling conditional random fields of spatially varying soil properties in slope reliability analysis. A saturated clay slope that is excavated at the site A of the west side highway in New York is investigated as an example to demonstrate the proposed approach. The short-term shear strength of the undrained clay is characterized by the undrained shear strength ($s_u$). The spatial variation of $s_u$ is depicted by a conditional random field incorporating site-specfiic data that are outcomes of field vane shear tests (VST) at this site (Asaoka and A-Grivas, 1982).

## 2 Sampling approach of conditional random field

The prior statistical information of the random fields of the soil properties is commonly assumed based upon the local experience, engineering judgment and knowledge revealed in geotechnical literature due to limited test data availability. It is well-known that lognormal distribution is particularly suitable for the soil parameters that cannot take on negative values since it ranges between zero and infinity, skewed to the low range. In this study, the undrained shear strength in the clay layer is assumed to follow lognormal distribution with parameters of $\lambda_{s_u} = \ln \mu_{s_u} - \xi_{s_u}^2/2$ and $\xi_{s_u} = \sqrt{\ln\left(1+\mathrm{COV}_{s_u}^2\right)}$, in which $\mu_{s_u}$ and $\mathrm{COV}_{s_u}$ are the mean and COV of $s_u$. Besides, the statistics of $s_u$ such as $\mu_{s_u}$ also exhibit uncertainties as reported in the literature (e.g., Cao et al., 2016; Huang et al., 2016; Papaioannou and Straub, 2017). The prior distribution of $\mu_{s_u}$ is also modeled by a lognormal distribution with parameters of $\lambda'_{\mu_{s_u}}$ and $\xi'_{\mu_{s_u}}$. The statistical information about $\mu_{s_u}$ or varying ranges of $\mu_{s_u}$ for different types of soils (e.g., cohesive soil, fine grained soil) can be obtained from the published literature (e.g., Rackwitz, 2000; Cao et al., 2016). Following Papaioannou and Straub (2017), the lower and upper bounds $\mu_{s_u}^{lower}$ and $\mu_{s_u}^{upper}$ are taken as the $p_1$ and $p_2$ quantiles of $\mu_{s_u}$, respectively. In this way, the $\lambda'_{\mu_{s_u}}$ and $\xi'_{\mu_{s_u}}$ can be obtained. The prior mean and standard deviation of $\mu_{s_u}$ are calculated by $\mu'_{\mu_{s_u}} = \exp\left(\lambda'_{\mu_{s_u}} + \xi'^2_{\mu_{s_u}}/2\right)$ and $\sigma'_{\mu_{s_u}} = \mu'_{\mu_{s_u}}\sqrt{\exp\left(\xi'^2_{\mu_{s_u}}\right)-1}$. It can be derived that the $\lambda_{s_u}$ follows a normal distribution with mean of $\mu'_{\lambda_{s_u}} = \lambda'_{\mu_{s_u}} - \xi_{s_u}^2/2$ and standard deviation of $\sigma'_{\lambda_{s_u}} = \xi'_{\mu_{s_u}}$.

As highlighted in Huang et al. (2016) and Papaioannou and Straub (2017), the posterior marginal distribution of $s_u$ at each spatial location, $s_u(q_i)$, is the predictive distribution of $s_u$. The posterior PDF of $s_u(q_i)$ can be evaluated by integrating out the distribution parameter $\lambda_{s_u}$ from the joint PDF of $s_u$ and $\lambda_{s_u}$ :

$$f''_{s_u}\left[s_u(q_i)\right] = \int_{-\infty}^{+\infty} f_{s_u}(s_u \mid \lambda_{s_u}) f'_{\lambda_{s_u}}(\lambda_{s_u}) d\lambda_{s_u} \tag{1}$$

A detailed derivation of Eq. (1) is given in Huang et al. (2016). It can be found that the posterior marginal distribution of $s_u(q_i)$ is also a lognormal distribution with parameters of $\mu'_{\lambda_{s_u}}$ and $\sqrt{\sigma'^2_{\lambda_{s_u}} + \xi_{s_u}^2}$. Therefore, $\ln s_u(\mathbf{q})$ follows a joint Gaussian distribution with mean vector of $\mu'_{\ln s_u}(\mathbf{q})$ and covariance matrix of $C'_{\ln s_u}(\mathbf{q},\mathbf{q})$, in which $\mathbf{q} = (q_1, q_2, \ldots, q_{n_e})^{\mathrm{T}}$, $q_i = (x_i, z_i)$ are the coordinates of the $i$-th spatial location and $n_e$ is the number of random field elements. The $i$-th element of $\mu'_{\ln s_u}(\mathbf{q})$ and the $(i, j)$-th entry of $C'_{\ln s_u}(\mathbf{q},\mathbf{q})$ are calculated by

$$\begin{cases} \mu'_{\ln s_u}(q_i) = \mu'_{\lambda_{s_u}} \\ C'_{\ln s_u}(q_i, q_j) = \sigma'^2_{\lambda_{s_u}} + \xi^2_{s_u} \rho'_U\left(q_i, q_j\right) \end{cases} \tag{2}$$

where $\rho'_U\left(q_i, q_j\right)$ is the autocorrelation coefficient between two standard normal variables at locations points $q_i$ and $q_j$, which is approximately equal to $\rho'_{s_u}\left(q_i, q_j\right)$.

The measurement errors resulting from imperfect measurement techniques, instruments or procedural controls in field tests are generally unavoidable (e.g., El-Ramly et al., 2002). To properly characterize the spatial variation of $s_u$, the measurement errors shall be considered when estimating the posterior statistics of the condition random field. The measurement result of $s_u$ at a given location $q_i^m = (x_i^m, z_i^m)$, $s_{u,i}^m$, can be related to a multiplicative measurement error $\varepsilon_i$ as follows (e.g., Straub and Papaioannou, 2015):

$$s_{u,i}^m = s_u(q_i^m)\varepsilon_i \tag{3}$$

where $s_u(q_i^m)$ is the simulated realization of $s_u$ at $q_i^m$; The $\varepsilon_i$, $i = 1, 2, \ldots, n_m$, are typically independent among tests and are assumed to follow the lognormal distributions with medians equal to one and constant standard deviations (e.g., DeGroot and Baecher, 1993; El-Ramly et al., 2002). $n_m$ is the number of measurements. With this assumption in mind, the likelihood function can be constructed as

$$L\left[s_u(\mathbf{q})\right] = \prod_{i=1}^{n_m} \phi\left[\frac{\ln s_{u,i}^m - \ln s_u(q_i^m)}{\sigma_{\ln \varepsilon_i}}\right] \tag{4}$$

where $\phi(.)$ is the PDF of the standard Gaussian variable; $\sigma_{\ln \varepsilon_i}$ is the standard deviation of $\ln \varepsilon_i$, which can be estimated by $\sigma_{\ln \varepsilon_i} = \sqrt{\ln\left(1 + \mathrm{COV}^2_{\varepsilon_i}\right)}$, in which $\mathrm{COV}_{\varepsilon_i}$ is COV of the $i$-th measurement error. The posterior mean vector of $\mu''_{\ln s_u}(\mathbf{q})$ and covariance matrix of $C''_{\ln s_u}(\mathbf{q}, \mathbf{q})$ incorporating the measurement uncertainties can be derived as (e.g., Stein, 1999)

$$\begin{cases} \mu''_{\ln s_u}(q_i) = \mu'_{\lambda_{s_u}} + C'_{\ln s_u}(q_i, \mathbf{q}^m)\left[C'_{\ln s_u}(\mathbf{q}^m, \mathbf{q}^m) + \sigma^2_{\ln \varepsilon_i}\right]^{-1}\left(\ln \mathbf{s}_{\mathbf{u}}^m - \mu'_{\lambda_{s_u}}\right) \\ C''_{\ln s_u}(q_i, q_j) = C'_{\ln s_u}(q_i, q_j) - C'_{\ln s_u}(q_i, \mathbf{q}^m)\left[C'_{\ln s_u}(\mathbf{q}^m, \mathbf{q}^m) + \sigma^2_{\ln \varepsilon_i}\right]^{-1} C'_{\ln s_u}(\mathbf{q}^m, q_j) \end{cases} \tag{5}$$

where $\mathbf{s}_{\mathbf{u}}^m = \left(s_{u,1}^m, s_{u,2}^m, \ldots, s_{u,n_m}^m\right)^{\mathrm{T}}$, $\mathbf{q}^m = (q_1^m, q_2^m, \ldots, q_{n_m}^m)^{\mathrm{T}}$. Based on the $\mu''_{\ln s_u}(\mathbf{q})$ and $C''_{\ln s_u}(\mathbf{q}, \mathbf{q})$, the posterior mean and standard deviation of $s_u$ at the given location $q_i$, $\mu''_{s_u}(q_i)$ and $\sigma''_{s_u}(q_i)$, can be obtained. Then, the midpoint method is adopted for sampling the conditional random field of $s_u$ (e.g., Liu et al., 2017).

## 3 Illustrative example: application to a saturated clay slope

A saturated clay slope under undrained conditions is investigated as an example in this section to illustrate the proposed approach. As shown in Figure 1, the slope has a height of $H$ = 9 m and a slope angle of 18.4°. The undrained clay is underlain by a firm stratum at 27 m below the top of slope and has a saturated unit weight of $\gamma_{sat}$ = 20 kN/m$^3$.

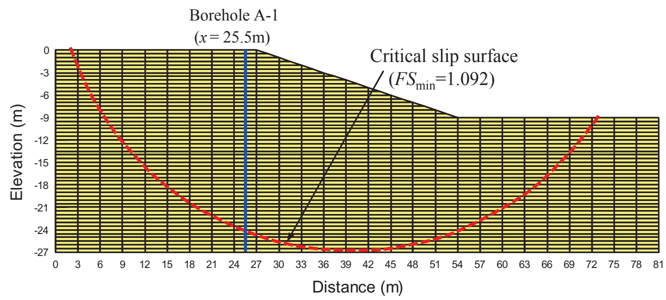



Figure 1. Random field mesh and slope stability analysis results.

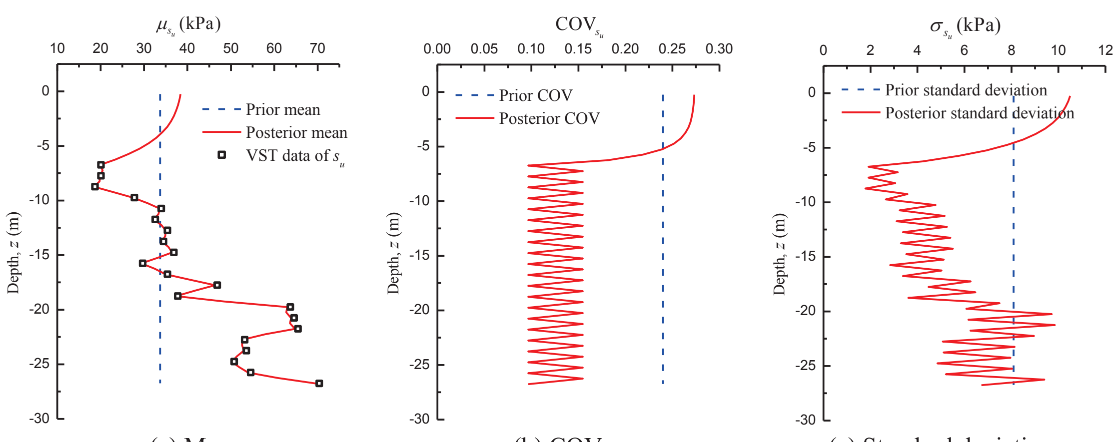
### 3.1 *Prior knowledge*

The prior statistical information of $s_u$ and measurement errors $\varepsilon_i$ is determined based on the literature and physical considerations (e.g. physical bounds) due to the limited test data. According to Rackwitz (2000), the value of $\mu_{s_u}$ is reported to be in the ranges of [10, 20 kPa], [20, 50 kPa] and [50, 100 kPa] for soft, stiff and very stiff inorganic plastic cohesive soils, respectively. The stiff inorganic plastic cohesive soil is taken as an example, the corresponding bounds of 20 and 50 kPa are treated as 10% and 90% quantiles of $\mu_{s_u}$, and a lognormal distribution is fitted. The resulting prior mean and standard deviation of $\mu_{s_u}$ are $\mu'_{\mu_{s_u}}$ = 33.71 kPa and $\sigma'_{\mu_{s_u}}$ = 12.45 kPa, respectively. As reported in Phoon and Kulhawy (1999a), the range of $\mathrm{COV}_{s_u}$ is [0.04, 0.44] and its mean value is 0.24 for the $s_u$ determined from field vane shear tests. On this basis, $\mathrm{COV}_{s_u}$ = 0.24 is selected. Additionally, the reference values for horizontal and vertical scales of fluctuation of undrained shear strength of clay are taken as $\lambda_h$ = 38 m and $\lambda_v$ = 3.8 m. A separable exponential autocorrelation function is adopted for characterization of the spatial correlation of $s_u$ (e.g., Li et al., 2016; Liu et al., 2017). The $\mathrm{COV}_{\varepsilon_i}$ for VST tests is in the range of [0.1, 0.2] as reported in Phoon and Kulhawy (1999b). For illustration, $\mathrm{COV}_{\varepsilon_i}$ = 0.1 is used. By this means, the prior knowledge of $s_u$ can be modeled by an unconditional random field with $\mu'_{s_u}$ = 33.71 kPa, $\mathrm{COV}_{s_u}$ = 0.24 and $\sigma'_{s_u}$ = 8.09 kPa.

### 3.2 *Data and results*

The 21 VST data of $s_u$ along the soil depth from Boreholes A-1 [see Table 1 in Asaoka and A-Grivas (1982)] are utilized to generate the conditional random field of $s_u$ and update the slope reliability. The data are collected from the site A of the west side highway in New York. The slope profile is then discretized into a total of 1224 4-noded quadrilateral elements with horizontal and vertical side lengths of 3.0 and 0.5 m, as shown in Figure 1. The short-term stability of the homogenous slope under undrained conditions is assessed using Bishop's simplified method. As a reference, the factor of safety ($FS$) evaluated for the prior mean of $s_u$ (i.e., $\mu'_{s_u}$ = 33.71 kPa) is 1.095. The location of the critical slip surface is plotted in Figure 1. A deep failure mechanism that is independent of the depth is observed. The failure event of the slope is defined as $FS$ being less than 1.0. Without measurements, the prior probability of slope failure based on the unconditional random field of $s_u$ is 0.366, as calculated by subset simulation with conditional probability $p_0 = 0.1$ and the number of samples at each level $N_l = 2000$ (Au and Beck, 2001).

To construct conditional random field of $s_u$, an arbitrary drilling location of $x = 25.5$ m is selected for Borehole A-1 as shown in Figure 1. The 21 VST data of $s_u$ obtained from Borehole A-1 are used. The posterior mean $\mu''_{s_u}(q_i)$ and standard deviation $\sigma''_{s_u}(q_i)$ of $s_u$ at each spatial location $q_i$ can be calculated using Eq. (5). Figure 2(a), (b) and (c) show the posterior means, COVs and standard deviations of $s_u$ along the depth ($x = 25.5$ m), respectively. In Figure 2(a), the posterior means at the measurement locations well matches the measured values and differ considerably from the prior mean. In Figure 2(b), the posterior COVs at the unmeasured locations that have same separations to the measurement locations along the depth are uniformly equal to 0.155, while those at the measurement locations are about 0.1, which is exactly equal to $\mathrm{COV}_{\varepsilon}$ . The posterior COV oscillates regularly around 0.125. The reason lies in that the posterior mean and standard deviation at one location only rely on the distances between the concerned location and the measurement locations [see Eq. (5)]. In Figure 2(c), the elements adjacent to the measurement locations have somewhat reduced posterior standard deviations. It is also interesting to observe that both the $\mu''_{s_u}$ and $\sigma''_{s_u}$ increase with the depth, which is in good agreement with the change trend of VST data with the depth. It indicates the proposed approach not only can make best use of the test data to reduce the uncertainties of $s_u$ in the estimation, but also can account for the non-stationary characteristics where the mean and standard deviation of $s_u$ increase with the depth. Based on the obtained posterior mean vector and covariance matrix of $s_u$, the realizations of the conditional random field of $s_u$ are generated by the midpoint method and mapped to the random field element mesh of slope profile. The posterior probability of failure is $9.5\times10^{-2}$, as calculated by subset simulation with $p_0 = 0.1$ and $N_l = 2000$, which is smaller than the prior probability of failure (i.e., $P_f = 0.366$). It implies that the slope reliability can be improved with the site investigation data incorporated in the generation of the conditional random field.




(a) Mean (b) COV (c) Standard deviation

Figure 2. Comparison of prior and posterior statistics of undrained shear strength along the depth ($x$ = 25.5 m).

## 4 Conclusions

This paper proposes an analytical approach for sampling conditional random field of undrained shear strength. A saturated clay slope has been investigated as an example to illustrate the proposed approach. The proposed approach not only can make best use of the limited site-specific data to learn the distribution of spatially varying undrained shear strength and estimate the posterior statistics of undrained shear strength analytically, but also can capture the depth-dependent nature of undrained shear strength. It can provide an effective means for properly characterizing the spatial variation of soil properties.

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