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A NEW ADAPTIVE SPARSE POLYNOMIAL CHAOS EXPANSION METHOD FOR STRUCTURAL RELIABILITY ANALYSIS

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To deal with the curse of dimensionality of polynomial chaos expansion for assessing reliability of structures with high stochastic dimensions, this paper proposes a novel non-intrusive algorithm based on a sparse partial least squares regression procedure. Firstly, an initial estimation of the expansion coefficients is obtained by performing partial least squares regression with all polynomials in the candidate set. Then, updated estimations of the expansion coefficients are iteratively obtained by eliminating a ratio of terms with smallest coefficients in absolute value. The quasi-optimal metamodel is selected by using a largest sparsity degree criterion which prevents overfitting. Next, the optimal metamodel is constructed with important inputs which are identified with an effective global sensitivity analysis procedure. Finally, structural reliability is effectively estimated by simulating the optimal metamodel. Numerical experiments verified that the proposed method outperforms the traditional counterpart in terms of computational efficiency and accuracy.

Keywords: Polynomial chaos expansion, Curse of dimensionality, Partial least squares regression, Structural reliability.

1 Introduction

Structural reliability analysis is of key importance in risk assessment and decision making. To reduce the heavy computational burden of simulation-based methods, metamodel-based methods are under active research. Many metamodeling procedures such as Kriging (Jiang and Li 2017), artificial neural networks (Dai et al. 2015), support vector machine (Dai and Cao 2017), polynomial chaos expansion (PCE) (Hawchar et al. 2017), low-rank tensor approximations (Konakli and Sudret 2016) have been proposed for constructing structural reliability algorithms. Among these procedures, PCE has gained much attention in recent years.

PCE projects the random structural response onto a Hilbert space spanned by orthonormal polynomials of the inputs (Xiu and Karniadakis 2002). The whole information about the probability distribution is encapsulated in the coefficients of the expansion. Due to mean-square convergence (Ernst et al. 2012) and fast convergence rates for smooth input-output relationships, PCE has become a powerful tool for uncertainty quantification in many territories. While mathematically elegant, PCE suffers from curse of dimensionality. To alleviate this problem, many approaches have been proposed in recent years such as stepwise regression (Abraham et al. 2017), least angle regression (Blatman and Sudret 2011), compressive sensing (Hu and Zhang 2017), support vector regression (Cheng and Lu 2018), D-MORPH regression (Cheng and Lu 2018a) and so on. These approaches have been verified to be capable of building metamodels with acceptable accuracy under small sample sizes.

This paper proposes a new PCE-based metamodeling method for structural reliability analysis. Different from the existing literature, a state-of-the-art regression method named partial least squares regression (PLSR) (Rosipal and Kramer 2006) is introduced to capture the latent low-dimensional structure of the model. Expression of the metamodel is greatly simplified with a largest sparsity degree criterion and an efficient global sensitivity analysis procedure, leading to accurate estimations of structural reliability. The proposed method is introduced in detail in section 2 and verified with two numerical examples in section 3.

2 The Proposed PCE-based Metamodeling Approach

The random structural response y (assumed to be unidimensional and zero-mean) can be expressed with the polynomial chaos expansion with order $p_{\max}$ in Eq.(1).

$$\mathrm{y} = \sum_{i=1}^{P} \beta_i \Psi_i(\boldsymbol{\xi}) \tag{1}$$

where β_i are expansion coefficients, Ψ_i are orthonormal polynomials, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)^{\mathrm{T}}$ is the vector of independent inputs and

$$P = \frac{(n + p_{\max})!}{n!\, p_{\max}!} - 1 \tag{2}$$

Under the framework of non-intrusive analysis, the PCE coefficients are originally computed with the ordinary least squares regression (OLSR) as Eq.(3).

$$\boldsymbol{\beta} = \mathbf{E}^{+}\mathbf{F} \tag{3}$$

where $\boldsymbol{\beta}$ is the vector of coefficients, $\mathbf{E}$ is the matrix whose each line contains values of polynomials at a sample point , $\mathbf{F}$ is the vector of structural response and "+" denotes the Moore-Penrose generalized inverse. Clearly, OLSR-PCE suffers from curse of dimensionality as shown in Eq.(2). Therefore, we propose a new method to build the metamodel for structural reliability analysis.

2.1 *Initialization*

Firstly, select $p_{\max}$ with *prior* knowledge about nonlinearity of the model. Then generate a set of N training samples with the Sobol quasi-random sampling scheme. Run high-fidelity simulations to get the centered response vector $\mathbf{F}$. Meanwhile, compute the polynomial matrix $\mathbf{E}$.

2.2 *Sparse partial least squares regression*

2.1.1 *Initial estimation of the expansion coefficients*

The initial estimation of the expansion coefficients is obtained with the PLSR. The idea of PLSR is to extract latent variables by iteratively solving the optimization problem in Eq.(4)

$$\begin{cases} \text{Find} \quad \mathbf{w}, \mathbf{c} \\ \text{s.t.} \quad \max\limits_{\substack{\|\mathbf{w}\|_2=1, \\ \|\mathbf{c}\|_2=1}} [\mathrm{Cov}^2(\mathbf{Ew}, \mathbf{Fc})] \end{cases} \tag{4}$$

where $\mathbf{w}$ and $\mathbf{c}$ are called loadings, $\mathbf{t} = \mathbf{Ew}$ and $\mathbf{u} = \mathbf{Fc}$ are the latent variables. Assume the relationship between is linear:

$$\mathbf{u} = b\mathbf{t} + \mathbf{H} \tag{5}$$

Let $\mathbf{u} = b\mathbf{t}$ and deflate $\mathbf{E}$ and $\mathbf{F}$ with OLSR as Eq.(6) and Eq.(7).

$$\mathbf{E} = \mathbf{E} - \mathbf{tp}^{\mathrm{T}} \tag{6}$$

$$\mathbf{F} = \mathbf{F} - \mathbf{uq}^{\mathrm{T}} \tag{7}$$

The times of iteration h is determined by minimizing the modified cross validation error in Eq.(8)

$$\varepsilon^*_{\mathrm{LOO(P)}} = \varepsilon_{\mathrm{LOO(P)}} \times \left(1 - \frac{h}{N}\right)^{-1} \left(1 + \mathrm{tr}((\mathbf{T}^{\mathrm{T}}\mathbf{T})^{-1})\right) \tag{8}$$

where $\varepsilon_{\mathrm{LOO(P)}}$ is the normalized pseudo leave-one-out cross validation error. The initial estimation is expressed with Eq.(9).

$$\boldsymbol{\beta} = \mathbf{E}^{\mathrm{T}}\mathbf{U}(\mathbf{T}^{\mathrm{T}}\mathbf{E}\mathbf{E}^{\mathrm{T}}\mathbf{U})^{-1}\mathbf{T}^{\mathrm{T}}\mathbf{F} \tag{9}$$

2.1.2 *Selection of the quasi-optimal metamodel*

Denote the length of $\boldsymbol{\beta}$ as P' and the retaining ratio as r. Retain the polynomials with the largest $[P' r]$ regression coefficients in absolute value. Then, PLSR is performed between $\mathbf{F}$ and the retained polynomials. This process is proceeded iteratively until $\varepsilon^*_{\mathrm{LOO(P)}}$ is larger than a prescribed threshold $\varepsilon^*_{\mathrm{LOO(P),th}}$. If this condition is not satisfied, retain the initial estimation.

2.1.3 *Construction of the optimal metamodel*

The optimal metamodel is constructed by using the sparse PLSR approach with the important inputs identified with variance-based global sensitivity analysis. Total Sobol index of each ξ_i is computed by a simple post-processing of the coefficients of the quasi-optimal metamodel, as shown in Eq.(10).

$$S_{\mathrm{T}i} = \frac{\sum\limits_{\boldsymbol{\alpha} \in \mathcal{F}_i^*} \beta_\alpha^2 E[\Psi_\alpha^2(\boldsymbol{\xi})]}{\sum\limits_{\boldsymbol{\alpha} \in \mathcal{A}} \beta_\alpha^2 E[\Psi_\alpha^2(\boldsymbol{\xi})]} \tag{10}$$

where

$$\mathcal{A} = \left\{ \boldsymbol{\alpha} \,\middle|\, \sum_{i=1}^{M} \alpha_i \leqslant p_{\max} \right\} \tag{11}$$

and

$$\mathcal{F}_i^* = \{\boldsymbol{\alpha} : \alpha_k > 0 \quad \forall k = 1, ..., M, \quad k = i\} \tag{12}$$

2.1.4 *Estimation of the failure probability*

Since the computational complexity of the optimal metamodel is much lower than that of the high-fidelity model, Monte Carlo Simulation (MCS) is feasible for computing the failure probability.

3 Case studies

Two different structures are exampled to test the accuracy and efficiency of the proposed method.

3.1 *Simply supported beam*

A simply supported beam subjected to a uniformly distributed load is shown in Figure 1

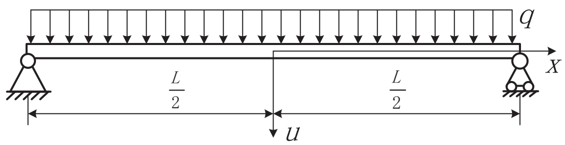

Figure 1 Configuration and loads of a simply supported beam

where $L = 3\text{m}$, inertial moment $I = 8\times10^{-6}\text{m}^4$ and $q = 13\text{kN/m}$. The elastic modulus is $E(x,\omega) = \exp(N(x,\omega))$ where $N(x,\omega)$ is a homogeneous Gaussian random field with correlation coefficient function $\rho(x,x') = \exp\left(-|x-x'|/l\right)$ where $l = 0.5\text{m}$. $N(x,\omega)$ is discretized with the first 40 components of Karhunen-Loeve expansion. The mean and coefficient of variation of elastic modulus are $\mu_E = 210\text{GPa}$ and $\delta_E = 0.2$, respectively. The failure event is defined as $u > 0.012\text{m}$. Let $p_{\max} = 3$ then we get $P = 12340$. Define $\gamma = N/P$ as the sample ratio. Let $\varepsilon^*_{\text{LOO(P),th}}$=0.001 .Comparison of the failure probabilities computed with OLSR-PCE and the proposed method is illustrated in Figure 2.

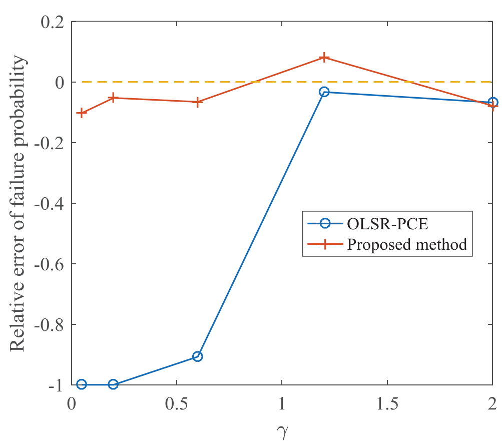

Figure 2 Comparison of failure probabilities in Example 1

The reference solution of failure probability in this example is 0.0024. The number of important inputs is two with the threshold of S_{Ti} is set to 0.018. It can be seen that the proposed method can get accurate results (error<10%) at $\gamma = 0.05(N = 617)$ while OLSR-PCE need $\gamma \geqslant 1.2$ to ensure accuracy. Thus, the computational gain factor is 24.

3.2 *Plain truss*

A plain truss subjected to vertical loads is shown in Figure 3

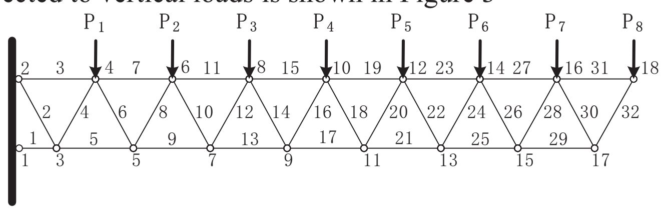

Figure 3 Configuration and loads of a plain truss

Each bar has elastic modulus E_i $(i = 1,...,32)$ and diameter 20mm. Distribution parameters of all the inputs are listed in Table 1. The quantity of interest is the vertical displacement of node 18, denoted as u. The failure event is defined as $u > 0.210\text{m}$.

Table 1 Distribution parameters of the inputs in Example 2

Variables	Distribution type	Mean	Standard deviation
E_i $(i = 1,...,32)$ (Pa)	Gaussian	2.0×10^{11}	3.0×10^{10}
P_1 (N)	Extreme 1	1.2×10^{4}	2.0×10^{3}
P_2 (N)	Extreme 1	1.0×10^{4}	1.5×10^{3}
P_3 (N)	Extreme 1	9.0×10^{3}	1.2×10^{3}
P_j $(j = 4,...,8)$ (N)	Extreme 1	8.0×10^{3}	1.0×10^{3}

Let $p_{\max} = 3$ then we get $P = 12340$. Let $\varepsilon^{*}_{\mathrm{LOO(P),th}}$=0.01 .Comparison of the failure probabilities computed with OLSR-PCE and the proposed method is illustrated in Figure 4.

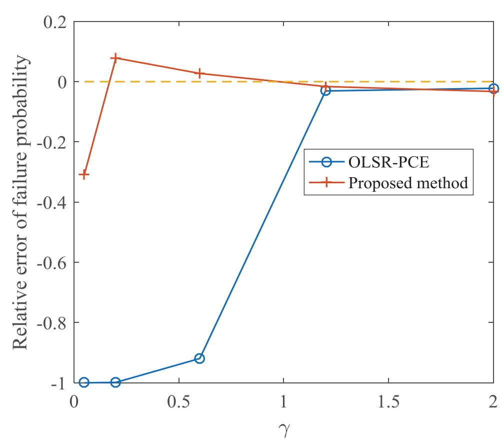

Figure 4 Comparison of failure probabilities in Example 2

All the inputs are transformed to standard Gaussian random variables. The reference solution of failure probability in this example is 0.0052. The number of important inputs is 14 with the threshold of $S_{\mathrm{T}i}$ is set to 0.005. It can be seen that the proposed method can get accurate results (error<10%) at $\gamma = 0.2$ ($N = 2468$) while OLSR-PCE need $\gamma \geqslant 1.2$ to ensure accuracy. Thus, the computational gain factor is 6 which is lower than the previous example since the effective stochastic dimension is much higher.

4 Conclusions

This paper proposes a new PCE-based metamodeling method for structural reliability analysis in high stochastic dimensions. The proposed method shed new light on PCE by extracting latent variables using PLSR. The proposed method has the potential of improving computational efficiency of structural reliability analysis, which is demonstrated by the results of numerical experiments.

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