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POINT-ESTIMATE METHOD FOR EVALUATING THE FAILURE PROBABILITY CONSIDERING THE UNCERTAINTIES OF DISTRIBUTION PARAMETERS

Zhao-Hui Lu[1,3], Pei-Pei Li[1] and Yan-Gnag Zhao[2, 3]

[1]*School of Civil Engineering, Central South University, 22 Shaoshannan Road, Changsha 410075, China. E-mail: luzhaohui@csu.edu.cn; lipeipei@csu.edu.cn*
[2]*Department of Architecture, Kanagawa University, 3-27-1 Rokkakubashi, Kanagawa-ku, Yokohama 221-8686, Japan. E-mail: Zhao@kanagawa-u.ac.jp*
[3]*Key Laboratory of Urban Security and Disaster Engineering of Ministry of Education, Beijing University of Technology, Beijing 100124, China*

Abstract

We investigate the evaluation of the failure probability considering the uncertainties of the distribution parameters of random variables. When these uncertainties are considered, the failure probability becomes a random variable that is referred to as the conditional failure probability. In the present paper, a point-estimate method based on univariate-dimension reduction integration is used to approximate the mean of the conditional failure probability. The simplicity, accuracy and efficiency of the proposed methodology for evaluating the failure probability considering the uncertainties of distribution parameters are numerically examined, where MCS is utilized for comparison. It is found that neglecting parameter uncertainties will lead to the failure probability being underestimated. Since the developed method can be realized only if the first few central moments of the basic random variables are known, it can be utilized even when the probability distributions of the basic random variables are unknown.

Keywords: Structural reliability; Parameter uncertainties; Conditional failure probability; Point-estimate method.

1 Introduction

A fundamental problem in structural reliability theory is the computation of the multifold probability integral (Sinozuka 1983; Zhao and Ono 2001)

$$P_f = \int_{G(\mathbf{X})\leq 0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{1}$$

where P_f is the probability of structural failure. In Eq. (1), $\mathbf{X} = [X_1, X_2, \dots, X_n]^T$ (where T denotes matrix transposition) is an n-dimensional vector of random variables representing uncertain quantities such as loads, material properties, geometric dimensions, and boundary conditions. Furthermore, $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function (PDF) of $\mathbf{X}$, $G(\mathbf{X})$ is the limit state function or performance function, and $G(\mathbf{X}) \leq 0$ is the domain of integration, which denotes the failure region of the structure.

One may regard Eq. (1) as a theoretical formulation of the structural reliability problem because the PDFs of the basic random variables (i.e., the components of $\mathbf{X}$ in Eq. (1)) are generally assumed to be known, and their distribution parameters in the PDFs are usually assumed to be certain. However, in practical engineering, one is faced with the problem of imperfect states of knowledge about such distributions. For example, the distribution parameters of the basic random variables involved in loads, environmental actions including chloride, temperature, oxygen, carbonation, moisture, and structural resistance are estimated from statistical data of limited sample size, and these distribution parameters may change as the amount of corresponding statistical data increases. All this results in uncertainties in the distribution parameters, and parameter uncertainties associated with the basic random variables in $\mathbf{X}$ lead to uncertainty in the calculated failure probability and in the associated reliability index (Der Kiureghian 1989).

In order to consider the uncertainties in the distribution parameters of a structural system, such as the mean and standard deviation of the basic random variables in $\mathbf{X}$, the distribution parameters are treated as a random vector $\mathbf{\Theta}$ in the Bayesian approach, whereby $f_{\mathbf{X}}(\mathbf{x})$ becomes a conditional distribution function $f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})$. Therefore, the conditional probability of failure becomes (Der Kiureghian 1989)

$$P_f(\mathbf{\Theta}) = \int_{G(\mathbf{X},\,\mathbf{\Theta})\le 0} f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})d\mathbf{x} \tag{2}$$

where $G(\mathbf{X},\mathbf{\Theta})$ expresses the performance function, $f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})$ is the joint PDF of $\mathbf{X}$ and $\mathbf{\Theta}$, and the conditional failure probability $P_f(\mathbf{\Theta})$ is a function of the distribution parameters $\mathbf{\Theta}$. Because the distribution parameters $\mathbf{\Theta}$ are uncertain, the conditional failure probability $P_f(\mathbf{\Theta})$ is also uncertain.

For vector $\mathbf{X}$ of the random variables in Eq. (2), whose joint PDF includes uncertain parameters $\mathbf{\Theta}$, the overall probability of failure is then defined as the expectation of the conditional failure probability $P_f(\mathbf{\Theta})$ over the outcome space of the uncertain parameters $\mathbf{\Theta}$, which can be formulated as

$$P_F = \int_{G(\mathbf{X},\,\mathbf{\Theta})\le 0} f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})d\mathbf{x}d\boldsymbol{\theta} \tag{3}$$

In most cases, Eq. (3) cannot be solved because of the difficulty in determining the explicit expression of the performance function $G(\mathbf{X},\mathbf{\Theta})$ and the joint PDF $f_{\mathbf{X},\mathbf{\Theta}}(\mathbf{x},\boldsymbol{\theta})$. This is because $\mathbf{\Theta}$ represents the distribution parameters of $\mathbf{X}$, but $\mathbf{X}$ is a function of $\mathbf{\Theta}$. However, the conditional failure probability of the structural system for given distribution parameter values $\mathbf{\Theta} = \boldsymbol{\theta}$ can be evaluated readily using state-of-the-art techniques such as the first- and second-order reliability methods, moment methods, and simulation methods (Ang and Tang 1984; Ditlevsen and Madsen 1996; Zhao and Ono 2001). Therefore, the overall probability of failure incorporating the uncertainties of the distribution parameters can be formulated generally as

$$P_F = \int_{\boldsymbol{\theta}} P_f(\mathbf{\Theta}) f_{\Theta}(\boldsymbol{\theta})d\boldsymbol{\theta} \tag{4}$$

where $P_f(\mathbf{\Theta})$ is the conditional probability of failure for a given $\mathbf{\Theta} = \boldsymbol{\theta}$ (which can be evaluated from state-of-the-art techniques), and $f_{\mathbf{\Theta}}(\boldsymbol{\theta})$ is the joint PDF of $\mathbf{\Theta}$.

An advanced first-order second-moment method, which developed from the first-order reliability method by introducing an auxiliary variable, for solving Eq. (4) has been proposed by Zhao and Jiang (1992), in which the effect of distribution parameter uncertainties on the overall probability of failure was discussed. An efficient analysis procedure was proposed by Hong (1996) to evaluate the overall probability of failure by using the point-estimate method to discretize the uncertain distribution parameters; the overall probability of failure was then obtained by weighting the conditional probability of failure at each discrete point. Later, Der Kiureghian (2008) derived a simple approximate formula by using the first-order approximation method to compute the mean of the conditional reliability index. Ang and De Leon (2005) developed Monte Carlo simulation (MCS) to solve this problem. All the methods mentioned above are assumed that the probability distributions of the basic random variables are known. However, due to the lack of statistical data, the probability distributions of some basic random variables are often unknown, and the probabilistic characteristics of these variables are often expressed using only statistical moments.

In the present paper, a point-estimate method based on univariate-dimension reduction integration is used to approximate the mean of the conditional failure probability including the basic random variables with unknown probability distributions. The simplicity, accuracy and efficiency of the proposed methodology for evaluating the failure probability considering the uncertainties of distribution parameters are numerically examined, where MCS is utilized for comparison.

2 Point-estimate method for evaluating the mean of the conditional failure probability

We note that the right-hand side of Eq. (4) represents the mean of the conditional failure probability $E[P_f(\mathbf{\Theta})]$. Therefore, the overall probability of failure incorporating the uncertainties of the distribution parameters is essentially the problem of estimating the mean of the conditional failure probability $P_f(\mathbf{\Theta})$. Rewriting Eq. (4) in standard normal space, we obtain

$$P_F = E[P_f(\mathbf{\Theta})] = \int_{\mathbf{u}} P_f[T^{-1}(\mathbf{u})]\phi(\mathbf{u})d\mathbf{u} \tag{5a}$$

where $\phi(\mathbf{u})$ denotes the PDF of each standard normal variable; and $T^{-1}(\mathbf{u})$ denotes the inverse normal transformation which can realized by using third-moment transformation (Zhao and Ono 2000a; Zhao et al. 2006; Lu et al. 2017) as follows.

$$x = T^{-1}(u) = \mu_x + \sigma_x\left[\frac{\alpha_{3x}}{6}(u^2-1) + \sqrt{1-\frac{\alpha_{3x}^2}{18}}u\right] \tag{5b}$$

Eq. (5) gives the mean of the conditional failure probability $P_f(\mathbf{\Theta})$, which is a function of the random vector $\mathbf{\Theta}$. In practice, the integral in Eq. (5) cannot be evaluated analytically because of its high dimensionality and the complicated integration required. To avoid this problem, we use the point-estimate method (Zhao and Ono 2000b) to solve Eq. (5), i.e., we evaluate the mean of $P_f(\mathbf{\Theta})$, which is one of the moments of function $P_f(\mathbf{\Theta})$.

Using the standard point estimate, the mean of $P_f(\mathbf{\Theta})$ (i.e., P_F) is estimated as

$$P_F = E[P_f(\mathbf{\Theta})] = \sum\prod_{i=1}^{n} P_{ci}\left\{P_f\left[T^{-1}(u_{c1}, \ldots, u_{ci}, \ldots, u_{cn})\right]\right\} \tag{6}$$

where n is the dimension of random vector $\mathbf{\Theta}$, c is a distinct combination of n items from group $[1, 2, \ldots, m]$, m is the number of estimating points, ci is the ith item of c, u_{ci} is the cith estimating point, and P_{ci} is the weight corresponding to u_{ci} ; and Σ is the sum of the calculations for each combination.

Because all distinct combinations have to be considered, m^n function calls are required to compute $P_f(\mathbf{\Theta})$. Therefore, the computations involved in Eq. (6) can be massive if n is large. To avoid this problem, we need to adopt dimension-reduction integration (Rahman and Xu 2004). Because only the first-order moment (i.e., the mean of $P_f(\mathbf{\Theta})$) is considered, the univariate-dimension reduction method (Zhao and Ono 2000b; Rahman and Xu 2004) is used here. The function $P_f(\mathbf{\Theta})$ may then be approximated by $P_f^*(\mathbf{\Theta})$ as follows:

$$P_f(\mathbf{\Theta}) \cong P_f^*(\mathbf{\Theta}) = \sum_{i=1}^{n}\left[P_{fi} - P_f(\boldsymbol{\mu})\right] + P_f(\boldsymbol{\mu}) = \sum_{i=1}^{n} P_{fi} - (n-1)P_f(\boldsymbol{\mu}) \tag{7}$$

where

$$P_{fi} = P_f(\mathbf{\Theta}_i) = P_f[T^{-1}(\mathbf{U}_i)] \tag{8}$$

and $\boldsymbol{\mu}$ represents the vector in which all the random variables take their mean values. In addition, $P_f(\boldsymbol{\mu})$ is a constant because it is the function of the mean of each random variable. Furthermore, we have $\mathbf{\Theta}_i = [\mu_1, \ldots, \mu_{i-1}, \theta_i, \mu_{i+1}, \ldots, \mu_n]^T$ and $\mathbf{U}_i = [u_{\mu 1}, \ldots, u_{\mu i □□}, u_i, u_{\mu i+1}, \ldots, u_{\mu n}]^T$, where $u_{\mu k}$, □$k = 1, \ldots, n$ except i is the kth value of u_μ, which is the vector in u space corresponding to $\boldsymbol{\mu}$. Finally, P_{fi} is a function of only u_i for specific $P_f{}^*(\mathbf{\Theta})$. For independent random variables $\mathbf{\Theta}$, P_{fi} can be expressed simply as

$$P_{fi} = P_f\left(\mu_1, \ldots, \mu_{i-1}, \theta_i, \mu_{i+1}, \ldots, \mu_n\right) = P_f\left[\mu_1, \ldots, \mu_{i-1}, T^{-1}(u_i), \mu_{i+1}, \ldots, \mu_n\right] \tag{9}$$

Observe that u_i ($i = 1, \ldots, n$) are independent and P_{fi} is a function of only u_i; therefore, P_{fi}, $i = 1, \ldots, n$ are also independent. Hence, the mean of $P_f{}^*(\mathbf{\Theta})$, i.e., the mean of the conditional failure probability, can be expressed as

$$P_F = E[P_f(\mathbf{\Theta})] \cong E[P_f^*(\mathbf{\Theta})] = \sum_{i=1}^{n} \mu_{P_{fi}} - (n-1)P_f(\boldsymbol{\mu}) \tag{10}$$

where $\mu_{P_{fi}}$ is the mean value of P_{fi} and can be point-estimated from

$$\mu_{P_{fi}} = E\left(P_{fi}\right) = E\left\{P_f\left[T^{-1}(\mathbf{U}_i)\right]\right\} = \sum_{k=1}^{m} P_k P_f\left[T^{-1}(u_{ik})\right] \tag{11}$$

where $u_{i1}, u_{i2}, \ldots, u_{im}$ are the estimating points of random variable u_i, and $P_1, P_2, \ldots, P_m$ are the corresponding weights.

The estimating points u_{ik} and their corresponding weights P_k can be readily obtained as

$$u_{ik} = \sqrt{2}x_k, \quad P_k = \frac{w_k}{\sqrt{\pi}} \tag{12}$$

where x_k and w_k are the abscissas and weights, respectively, for Hermite integration with the weight function $\exp(-x^2)$, as given in Abramowitz and Stegun (1972). For a seven-point estimate ($m = 7$) in standard normal space (Zhao and Ono 2000b), we have the following:

$$u_{i1} = -u_{i7} = -3.7504397, \quad P_1 = P_7 = 5.48269\times10^{-4} \tag{13}$$

$$u_{i2} = -u_{i6} = -2.3667594, \quad P_2 = P_6 = 3.07571\times10^{-2} \tag{14}$$

$$u_{i3} = -u_{i5} = -1.1544054, \quad P_3 = P_5 = 0.2401233 \tag{15}$$

$$u_{i4} = 0, \quad P_4 = 0.4571427 \tag{16}$$

3 Numerical examples

Considers a bar subjected to tensile stress. The bar fails if the applied load exceeds the tensile strength of the bar, and the performance function is expressed simply as

$$G(\mathbf{X}) = R - S \tag{17}$$

where R is the resistance of the bar and S is the applied load.

Assume that R and S are independent random variables with means μ_R and μ_S, standard deviations σ_R and σ_S, and skewness of $\alpha_{3R} = \alpha_{3S} = 0$, respectively. For this example, the third-moment reliability index is available as a closed-form equation (Zhao et al. 2006)

$$\beta_{3M} = \beta_{2M} = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}} \tag{18}$$

and the failure probability is

$$P_f = \Phi(-\beta_{3M})$$

Table 1. Probabilistic information about the distribution parameters

Parameter	Distribution	Mean	Standard deviation
μ_R	Lognormal	50	2
μ_S	Lognormal	35	2
σ_S	Lognormal	5	1
σ_R	Lognormal	7	1.4

Here, the distribution parameters μ_R, μ_S, σ_R, and σ_S are assumed to be random variables, and their probabilistic information is listed in Table 1. According to Eq. (4), the overall failure probability can be formulated by

$$P_F = \int_{\boldsymbol{\theta}} P_f(\boldsymbol{\Theta}) f_{\Theta}(\boldsymbol{\theta}) d\boldsymbol{\theta} = \int_{\mu_R,\mu_s,\sigma_R,\sigma_s} P_f(\mu_R,\mu_s,\sigma_R,\sigma_s) f(\mu_R,\mu_s,\sigma_R,\sigma_s) d\mu_R d\mu_s d\sigma_R d\sigma_s \quad (19)$$

in which

$$P_f(\boldsymbol{\Theta}) = P_f(\mu_R,\mu_s,\sigma_R,\sigma_s) = \Phi\left(-\frac{\mu_R-\mu_S}{\sqrt{\sigma_R^2+\sigma_S^2}}\right) \quad (20)$$

According to Eq. (7), the conditional failure probability $P_f(\Theta)$ can be approximated as

$$P_f(\boldsymbol{\Theta}) \cong P_f^*(\boldsymbol{\Theta}) = \sum_{i=1}^{4} P_{fi} - 3P_f(\boldsymbol{\mu}) \quad (21)$$

where

$$P_{f1} = P_f(\mu_R) = \Phi\left(-\frac{\mu_R-35}{\sqrt{74}}\right),\quad P_{f2} = P_f(\mu_S) = \Phi\left(-\frac{50-\mu_S}{\sqrt{74}}\right), P_{f3} = P_f(\sigma_R) = \Phi\left(-\frac{15}{\sqrt{\sigma_R^2+5^2}}\right),$$

$$P_{f4} = P_f(\sigma_S) = \Phi\left(-\frac{15}{\sqrt{7^2+\sigma_S^2}}\right),\quad P_f(\boldsymbol{\mu}) = \Phi\left(-\frac{50-35}{\sqrt{7^2+5^2}}\right) = 0.0406$$

Using a seven-point estimate in standard normal space as given by Eqs. (13)–(16), the estimating points of $P_f(\mu_R)$ in original space can be obtained with the aid of Eq.(5b), and are given as follows:

$$\mu_{R1} = 43.003,\ \mu_{R2} = 45.449,\ \mu_{R3} = 47.706,\ \mu_{R4} = 49.96$$
$$\mu_{R5} = 52.320,\ \mu_{R6} = 54.919,\ \mu_{R7} = 58.043.$$

Therefore, the mean of P_{f1}, $\mu_{P_{f1}}$, is readily obtained as

$$\mu_{P_{f1}} = \sum_{k=1}^{7} P_k P_f(\mu_{Rk}) = 4.467\times10^{-2}$$

Similarly, the means of P_{f2}, P_{f3}, and P_{f4} are obtained as $\mu_{P_{f2}} = 4.477\times10^{-2}$, $\mu_{P_{f3}} = 4.245\times10^{-2}$, and $\mu_{P_{f4}} = 4.160\times10^{-2}$, respectively.

Therefore, according to Eq. (10), the overall probability of failure, i.e., the mean of the conditional failure probability, is readily estimated as

$$P_F = E[P_f(\boldsymbol{\Theta})] \cong \sum_{i=1}^{4} \mu_{P_{fi}} - 3P_f(\boldsymbol{\mu}) = 5.168\times10^{-2}$$

Using an MCS with 1,000,000 samples, the overall probability of failure, i.e., the mean of the conditional failure probability (Eq. (20)), is obtained as 5.123×10^{-2}. One can see that the result obtained by using the proposed method is close to that of the MCS. From Eq. (18), the failure probability without considering the parameter uncertainties is readily obtained as 4.06×10^{-2}, which is less than the mean of the conditional failure probability when considering the parameter uncertainties, i.e., the failure probability is underestimated or the structural reliability is overestimated.

4 Conclusions

A point-estimate method based univariate dimension-reduction integration was developed to evaluate the failure probability considering the uncertainties of distribution parameters. It is found that neglecting parameter uncertainties will lead to the failure probability being underestimated. Since the developed method can be realized only if the first few central moments of the basic random variables are known, it can be utilized even when the probability distributions of the basic random variables are unknown.

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