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OUTLIER ANALYSIS TO DETECT DAMAGE IN A STEEL TRUSS BRIDGE

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This study presents an outlier analysis to detect damage in highway bridges. A damage indicator automatically derived from a set of multivariate linear system models is proposed. The damage indicator evaluates a stochastic distance between a set of healthy bridge data and unknown data. Statistical hypothesis testing based on a probability distribution of the damage indicator was applied for damage detection. A damage experiment on an actual steel truss bridge was conducted to verify validity of the proposed outlier analysis for damage detection. Observations demonstrated that the proposed outlier analysis detected three damage patterns successfully, and even identified the sensors near the artificial damage.

Keywords: Bayesian Inference, damage detection, outlier analysis, vibration monitoring.

1 Introduction

Techniques of structural health monitoring (SHM) based on vibration measurements have been studied to reduce cost relevant to the visual inspection. Changes in structural integrity of bridges engender changes in their modal properties that are identifiable from vibration data (Zhang 2007). For bridge health monitoring, the ambient vibration test appears to be a more convenient method than the forced vibration test because natural excitations require no traffic control. In ambient vibration tests conducted on actual bridges, however, identified modal properties can be contaminated by unknown noises. To avoid the noise on the identified properties, existing studies have developed a damage indicator that defined directly from a mechanical system model representing the bridge vibration. Nair et al. (2006) investigated damage sensitive-feature consisting of univariate autoregressive (UAR) coefficients for a model building. Kim et al. (2016) verified the validity of the damage-sensitive feature from a field experiment conducted on an actual steel truss bridge with artificial damage. To cope with difficulties in decision-making for bridge maintenance, Goi and Kim (2016) have investigated a hypothesis-testing-based damage detection using a vector autoregressive (VAR) model. However, the classical hypothesis testing based on the Neyman-Pearson decision rule tends to lead the type I error, i.e., incorrect rejection of the null hypothesis. This kind of errors is one of possible matters in decision-making.

This study investigates a statistical damage detection method by means of Bayesian statistics. Firstly, the VAR model provides a likelihood function for observed bridge acceleration, and thus the Bayesian inference method for the VAR model provides the posterior distribution of the parameters of the VAR model. Accordingly, time series of acceleration acquired from a bridge under healthy condition provide a reference model of the bridge vibration. Secondly, based on the posterior distribution, damage-sensitive features from the parameters of the VAR model related to the structural properties of the bridge are extracted utilizing singular value decomposition (SVD). Thirdly, Bayesian hypothesis test is formulated to distinguish whether a newly observed time series is acquired from healthy bridge or not. This study adopts ratio of marginal likelihood of two different hypotheses, called Bayes factor (BF) (Kass and Raftery 1995), as a damage indicator. Feasibility of the proposed method is discussed utilizing field experiment data in which three different levels of artificial damage are introduced to two tension members of an actual steel truss bridge.

2 Bayesian Inference

Let $\boldsymbol{y}(k) \in \mathrm{R}^{m\times 1}$ denote a column vector of the discrete time series of measured acceleration whose components respectively correspond to *m* measurement points. The following VAR model approximately models the time series obtained from a linear structural system excited by the white noise with sufficient model order *P* (He and De Roeck 1997).

$$\boldsymbol{y}(k) = \sum_{i=1}^{P} A_i \boldsymbol{y}(k-i) + \boldsymbol{e}(k) \tag{1}$$

where, $A_i \in \mathrm{R}^{m\times m}$ denotes the *i*-th AR coefficient matrix and $\boldsymbol{e}(k) \in \mathrm{R}^{m\times 1}$ denotes the white noise vector. Focusing on *j*-th row in Eq. (1), the following regressive model is obtained.

$$y^{\{j\}}(k) = \sum_{i=1}^{P} \boldsymbol{a}_i^{\{j\}} \boldsymbol{y}(k-i) + e^{\{j\}}(k) \tag{2}$$

where, $y^{\{j\}}(k)$ and $e^{\{j\}}(k)$ respectively represent *j*-th element of $\boldsymbol{y}(k)$ and $\boldsymbol{e}(k)$, and $\boldsymbol{a}_i^{\{j\}} \in \mathrm{R}^{1\times m}$ represents *j*-th row of A_i. Assuming elements of $\boldsymbol{e}(k)$ are statistically independent each other and following Gaussian distribution with expectation 0, then $y^{\{j\}}(k)$ also follows the Gaussian distribution with the expectation $\sum_{i=1}^{P} \boldsymbol{a}_i^{\{j\}} \boldsymbol{y}(k-i)$. Letting $t_k = y^{\{j\}}(k)$, $\boldsymbol{w} = [\boldsymbol{a}_1^{\{j\}}, \dots, \boldsymbol{a}_P^{\{j\}}]^{\mathrm{T}} \in \mathrm{R}^{mP\times 1}$ and $\boldsymbol{\phi}_k = [\boldsymbol{y}(k-1)^{\mathrm{T}}, \dots, \boldsymbol{y}(k-P)^{\mathrm{T}}]^{\mathrm{T}} \in \mathrm{R}^{mP\times 1}$ for simplicity, the probability distribution function (PDF) of *t* is given as follows.

$$\mathrm{p}(t_k|\boldsymbol{\phi}_k, \boldsymbol{w}, \beta) = \mathrm{N}(t_k|\boldsymbol{w}^T\boldsymbol{\phi}_k, \beta^{-1}) \tag{3}$$

where, $\mathrm{N}(x|\mu, \sigma^2)$ denotes PDF of *x* following Gaussian distribution with expectation μ and variance σ^2, and β denotes the precision parameter of the regression, which is the inverse of the variance of the noise term $e^{\{j\}}(k)$. Assuming *n* samples of t_k and $\boldsymbol{\phi}_k$ are observed, and letting $\boldsymbol{t} = [t_1, \dots, t_n] \in \mathrm{R}^{n\times 1}$ and $\Phi = [\boldsymbol{\phi}_1 \ \dots \boldsymbol{\phi}_n]^{\mathrm{T}} \in \mathrm{R}^{n\times mP}$, then the likelihood function for the parameters $\boldsymbol{w}$ and β is defined as follows.

$$\mathrm{p}(\boldsymbol{t}|\Phi, \boldsymbol{w}, \beta) = \prod_{k=1}^{n} \mathrm{N}(t_k|\boldsymbol{w}^T\boldsymbol{\phi}_k, \beta^{-1}) \tag{4}$$

Bayesian theorem provides the posterior joint PDF for $\boldsymbol{w}$ and β as shown in Eq. (5).

$$\mathrm{p}(\boldsymbol{w}, \beta|\boldsymbol{t}) = \mathrm{p}(\boldsymbol{t}|\boldsymbol{w}, \beta)\mathrm{p}(\boldsymbol{w}, \beta)\mathrm{p}(\boldsymbol{t})^{-1} \tag{5}$$

where, Φ is omitted from above Eq. (5) for simplicity. The $\mathrm{p}(\mathbf{w}, \beta)$ stands for a prior joint PDF for $\boldsymbol{w}$ and β. The $\mathrm{p}(\boldsymbol{t})$ is a constant in manner of Bayesian inference, and therefore the posterior PDF $\mathrm{p}(\boldsymbol{w}, \beta|\boldsymbol{t})$ is obtained only from the observed data $\boldsymbol{t}$ and the prior PDF $\mathrm{p}(\boldsymbol{w}, \beta)$.

This study adopts the following prior that is conjugate to the likelihood function in Eq. (4).

$$\mathrm{p}(\boldsymbol{w}, \beta) = \mathrm{N}(\boldsymbol{w}|\boldsymbol{m}, \beta^{-1}L^{-1})\mathrm{Gam}(\beta|a, b) \tag{6}$$

Therein, the parameters $\boldsymbol{m}, L, a$ and b are hyperparameters of the prior PDF, which determine the functional properties of the PDF. $\mathrm{Gam}(\lambda|a, b)$ denotes PDF following Gamma distribution defined in Eq. (7) in which $\Gamma(\cdot)$ denotes the Gamma function.

$$\mathrm{Gam}(\lambda|a, b) = \frac{1}{\Gamma(a)} b^a \lambda^{a-1} \exp(-b\lambda) \tag{7}$$

In general, a conjugate prior provides a posterior distribution that has the same functional form as the prior (Bishop, 2006); thus Eqs. (4)–(6) lead to the following posterior distribution with hyperparameters, $\boldsymbol{m}'$, L', a' and b'.

$$\mathrm{p}(\boldsymbol{w},\beta|\boldsymbol{t}) = \mathrm{N}(\boldsymbol{w}|\boldsymbol{m}',\beta^{-1}L'^{-1})\mathrm{Gam}(\beta|a',b') \tag{8}$$

where, $\boldsymbol{m}' = L'^{-1}(L\boldsymbol{m} + \Phi^{\mathrm{T}}\boldsymbol{t})$, $L' = L + \Phi^{\mathrm{T}}\Phi$, $a' = a + n/2$, and $b' = b + \frac{1}{2}\left(\|\boldsymbol{t} - \Phi\boldsymbol{m}'\|^2 + (\boldsymbol{m} - \boldsymbol{m}')^T L(\boldsymbol{m} - \boldsymbol{m}')\right)$.

If any information related to the prior distribution is given beforehand, we can utilize non-informative prior to avoid arbitrary assumptions (Jeffreys 1946). Numerically, the posterior PDF complying with the non-informative prior is obtained by substituting $[L]_{ij}$=0 (i, j = 1… mP), a=0 and b=0 to the hyperparameters in Eq. (8). Once the posterior PDF is obtained from the non-informative prior, the posterior PDF is used as the prior PDF for the newly observed time series. Therefore, $\boldsymbol{m}'$, L', a' and b' of the posterior PDF are updated once a new data is obtained.

3 Feature Extraction

Let D_{ref} denote a reference dataset from acceleration time series of a healthy bridge, then the hyperparameters, $\boldsymbol{m}_{ref}$, L_{ref}, a_{ref} and b_{ref}, representing the posterior distribution $\mathrm{p}\left(\boldsymbol{w},\beta\middle|D_{ref}\right)$ are estimated by hyperparameters, $\boldsymbol{m}'$, L', a' and b'. Updated L' produces real, symmetric and positive definite matrix L_{ref}; thus the SVD of the hyperparameter L_{ref} is given as follows.

$$L_{ref} = U\Lambda U^{\mathrm{T}} = [U_1\ U_2]\begin{bmatrix}\Lambda_1 & O\\ O & \Lambda_2\end{bmatrix}\begin{bmatrix}U_1^{\mathrm{T}}\\ U_2^{\mathrm{T}}\end{bmatrix} \tag{9}$$

where, $\Lambda \in \mathrm{R}^{mP\times mP}$ is the diagonal matrix of the singular values and $U \in \mathrm{R}^{mP\times mP}$ is the orthogonal matrix of the singular vectors. $\Lambda_1 \in \mathrm{R}^{q\times q}$ and $U_1 \in \mathrm{R}^{mP\times q}$ respectively represent the q largest singular values and the corresponding singular vectors; Λ_2 and U_2 represent remaining singular values and singular vectors.

Let $\widetilde{\boldsymbol{w}}$ denote the orthogonal transformation of $\boldsymbol{w}$ such that $\widetilde{\boldsymbol{w}} = U^{\mathrm{T}}\boldsymbol{w}$; and $\widetilde{\boldsymbol{w}}_1 = U_1^{\mathrm{T}}\boldsymbol{w}$ and $\widetilde{\boldsymbol{w}}_2 = U_2^{\mathrm{T}}\boldsymbol{w}$. Then Eq. (8) leads to the following posterior distribution of $\widetilde{\boldsymbol{w}}_1, \widetilde{\boldsymbol{w}}_2$ and β utilizing the relationship, $\mathrm{N}(\widetilde{\boldsymbol{w}}|\widetilde{\boldsymbol{m}},\beta^{-1}\Lambda^{-1}) = \mathrm{N}(\widetilde{\boldsymbol{w}}_1|\widetilde{\boldsymbol{m}}_1,\beta^{-1}\Lambda_1^{-1})\mathrm{N}(\widetilde{\boldsymbol{w}}_2|\widetilde{\boldsymbol{m}}_2,\beta^{-1}\Lambda_2^{-1})$.

$$\mathrm{p}\left(\widetilde{\boldsymbol{w}},\beta\middle|D_{ref}\right) = \mathrm{N}(\widetilde{\boldsymbol{w}}_1|\widetilde{\boldsymbol{m}}_1,\beta^{-1}\Lambda_1^{-1})\mathrm{N}(\widetilde{\boldsymbol{w}}_2|\widetilde{\boldsymbol{m}}_2,\beta^{-1}\Lambda_2^{-1})\mathrm{Gam}\left(\beta\middle|a_{ref},b_{ref}\right) \tag{10}$$

where, $\widetilde{\boldsymbol{m}} = U^{\mathrm{T}}\boldsymbol{m}_{ref}$, $\widetilde{\boldsymbol{m}}_1 = U_1^{\mathrm{T}}\boldsymbol{m}_{ref}$ and $\widetilde{\boldsymbol{m}}_2 = U_2^{\mathrm{T}}\boldsymbol{m}_{ref}$. In the posterior distribution, the parameter $\widetilde{\boldsymbol{w}}_1$ has less variations compared to the $\widetilde{\boldsymbol{w}}_2$, hence the $\widetilde{\boldsymbol{w}}_1$ represents the parametric subspace that is certainly inferable from the observation.

This study presumes that the $\widetilde{\boldsymbol{w}}_1$ is related to modal properties of the bridge, and conducts the hypothesis testing to detect changes in the $\widetilde{\boldsymbol{w}}_1$. Let D_{test} denote a newly observed test dataset for the hypothesis testing, the BF for a null hypothesis H_0 and an alternative hypothesis H_1 is defined as a ratio of their marginal likelihoods as follows (Kass and Raftery, 1995).

$$B = \frac{\mathrm{p}(D_{test}|\mathrm{H}_1)}{\mathrm{p}(D_{test}|\mathrm{H}_0)} = \frac{\int \mathrm{p}(D_{test}|\boldsymbol{\theta})\mathrm{p}(\boldsymbol{\theta}|\mathrm{H}_1)\mathrm{d}\boldsymbol{\theta}}{\int \mathrm{p}(D_{test}|\boldsymbol{\theta})\mathrm{p}(\boldsymbol{\theta}|\mathrm{H}_0)\mathrm{d}\boldsymbol{\theta}} \tag{11}$$

where, the parametric vector $\boldsymbol{\theta}$ is consists of $\widetilde{\boldsymbol{w}}$ and β. The null and alternative hypotheses respectively provide stochastic models representing healthy and damaged conditions of the bridge. The BF is a summary of the evidence provided by the observed dataset D_{test} in favor of a statistically modelled hypothesis as opposed to another. Kass and Raftery (1995) suggested interpreting the BF on the natural logarithm scale. For example, if $2\ln B$ is over 10, then the evidence against the null hypothesis H_0 is interpreted to be '*very strong*'. This study adopts the likelihood function $\mathrm{p}(D_{test}|\boldsymbol{\theta})$ given in Eq. (4), which is independent of the hypotheses. The hypotheses are thus modeled in the parametric distribution prior to observing dataset D_{test}, i.e., $\mathrm{p}(\boldsymbol{\theta}|\mathrm{H}_0)$ and $\mathrm{p}(\boldsymbol{\theta}|\mathrm{H}_1)$ in Eq. (11).

The null hypothesis representing the healthy condition is modeled as shown in Eq. (12) using the posterior PDF to the reference dataset D_{ref}, which is given in Eq. (10).

$$\mathrm{p}(\boldsymbol{\theta}|\mathrm{H}_0) = \mathrm{N}(\widetilde{\boldsymbol{w}}|\widetilde{\boldsymbol{m}}, \beta^{-1}\Lambda^{-1})\mathrm{Gam}\left(\beta\middle|a_{ref}, b_{ref}\right) \tag{12}$$

The alternative hypothesis represents that $\widetilde{\boldsymbol{w}}_1$ somehow changes due to damage in the bridge. However, an exact PDF for $\widetilde{\boldsymbol{w}}_1$ is undefinable because of uncertainty of damage itself. Kass and Raftery (1995) have mentioned that the Schwarz criterion (Schwarz, 1978) can be applied as a standard procedure when the priors are hard to set precisely. This study adopts the Schwarz criterion presuming only the $\widetilde{\boldsymbol{w}}_1$ is uncertain and the other parameters follow the PDF posterior to the D_{ref}. The Schwarz criterion produces following approximation of the marginal likelihood under the alternative hypothesis in the log scale.

$$\ln \mathrm{p}(D_{test}|\mathrm{H}_1) \approx \sup_{\widetilde{\boldsymbol{w}}_1 \in \mathrm{R}^{q\times 1}} \iint \mathrm{p}(D_{test}|\widetilde{\boldsymbol{w}}_1, \widetilde{\boldsymbol{w}}_2, \beta)\mathrm{p}\left(\widetilde{\boldsymbol{w}}_2, \beta\middle|D_{ref}\right)\mathrm{d}\widetilde{\boldsymbol{w}}_2\mathrm{d}\beta \; - \frac{q}{2}\ln n_{test} \tag{13}$$

where, n_{test} denotes number of data samples contained in the D_{test}. Let $\boldsymbol{m}'$, L', a' and b' respectively denote hyperparameters comprising the PDF posterior to both of D_{ref} and D_{test}, i.e., $\mathrm{p}\left(\boldsymbol{w}, \beta\middle|D_{ref}, D_{test}\right) = \mathrm{N}(\boldsymbol{w}|\boldsymbol{m}', \beta^{-1}L'^{-1})\mathrm{Gam}(\beta|a', b')$, and re-arranging Eqs. (11)–(13) provides the following BF in the log scale.

$$2\ln B = 2a' \ln \frac{b'}{b'-d} - \ln\det(U_1^{\mathrm{T}} L'^{-1} U_1 \Lambda_1) - q \ln n_{test} \tag{14}$$

$$d = \frac{1}{2}\left(\boldsymbol{m}' - \boldsymbol{m}_{ref}\right)^{\mathrm{T}} U_1 \{U_1^{\mathrm{T}}(L^{-1} - L'^{-1})U_1\}^{-1} U_1^{\mathrm{T}}\left(\boldsymbol{m}' - \boldsymbol{m}_{ref}\right) \tag{15}$$

4 Application to Real Bridge

4.1 *Damage experiment on real truss bridge*

Field experiments were conducted with a moving vehicle on an actual bridge. The target bridge for the field experiment is a single lane simply supported through-type steel Warren truss bridge as shown in Figure 1. The bridge has 59.2 m span length, 8 m maximum height, and 3.6 m width. The vehicle used for the experiment is a two-axle recreation vehicle with total weight of about 21 kN. During the experiment, all traffic was blocked except the load vehicle. Eight uniaxial accelerometers were installed on the deck of the bridge to measure vertical vibrations as presented in Figure 1b. The sampling rate of each sensor was set as 200 Hz.

Five scenarios were considered in this study as shown in Figure 1c. Initially, the INT scenario represents the intact bridge with no damage. For the DMG1 scenario, a half-cut damage was applied to the vertical truss member at the mid-span (see Figure 1b), and for the DMG2 scenario, a full cut damage was applied to the same member. After examining the DMG2 scenario, the damaged member was repaired, which is denoted as the RCV scenario. Finally, for the DMG3 scenario, full cut was applied in a vertical member at 5/8th-span (see Figure 1b) after examining the RCV scenario. Each experiment was conducted under the vehicle running at about 40 km/h.

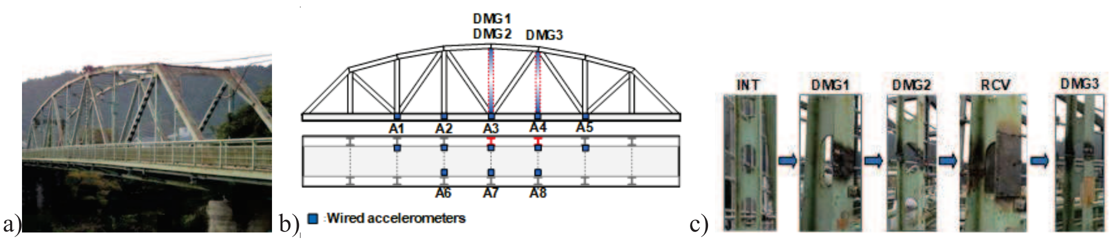

Figure 1. The target bridge: a) photo; b) sensor deplyment; c) damage scenarios.

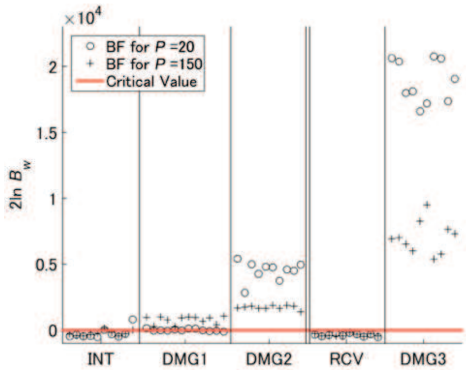

Figure 2. Outlier analysis with global BFs.

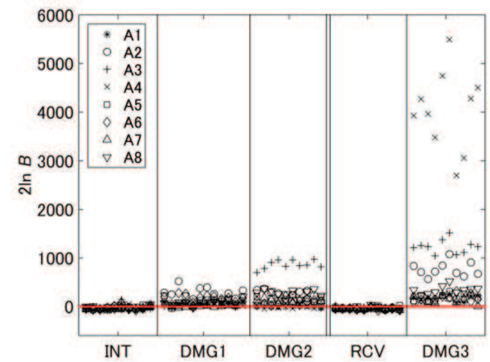

Figure 3. Outlier analysis with local BFs (P =150).

The vehicle was accelerated before entering to the bridge and the passing speed was kept to 40 km/h as possible by the speedometer. Considering variation in the passing speed, however, the bridge vibration under the passing vehicle was measured ten times for each of the scenarios. It is noteworthy that the proposed method is for utilizing ambient vibrations of bridges under varying vehicles with varying speeds, and is expected to detect the outliers from the sets of ambient vibration data although quality of the outlier analysis might be varied.

4.2 *Outlier analysis for damage detection*

The VAR order P and the number of the damage sensitive features q are predefined. The orders are determined by following steps. Firstly, assuming all of the parameters are uncertain, the Schwarz criterions evaluates the relevant VAR orders using the reference dataset; the VAR order providing highest Schwarz criterion is applied as the P for the hypothesis testing. Secondly, using the hyperparameters reproduced by the VAR model with P order, the q is determined by finding distinct singular values provided in the Λ. P=150 and q=10 were adopted in this study.

The time series acquired from the INT scenario were adopted as the reference dataset for DMG1 and DMG2 scenarios. For DMG3, the time series from RCV scenario was adopted as the reference dataset because the modal characteristics of the bridge might be changed after repairing the damaged member. BFs are calculated for each of the time series. In this study, the leave-one-out cross validation (CV) technique (Bishop, 2006) is applied to assess the validity of the BF; for the INT and RCV scenarios, the CV samples are evaluated using each of the time series as a test dataset and the remaining nine time-series under the same scenario as a reference dataset. $2\ln B$ was adopted as a test statistic, and a pre-defined threshold for the hypothesis testing is fixed to $2\ln B = 0$, where the model evidences for the null and the alternative hypotheses are equivalent.

From a pair of the reference and test datasets, mBFs are obtainable as each of the measurement points provides its regressive model. Assuming each of the BFs is independent, the product of those BFs represents the BF for the whole observation (hereafter, global BF). In Figure 2 the global BFs for P =20 and P =150 are shown, which shows higher P resulted in more sensitive outlier analysis. Figure 2 also shows that all of the damage scenarios resulted in higher BFs than the threshold such as higher than $2\ln B = 10$, which represents '*very strong*' evidence against the null hypothesis. Apparently, Figure 2 also indicates severity of damage of the member. Figure 3 that describes the BFs for each of the measurement points (called local BF) for P = 150 suggests possibility of damage localization for DMG2 and DMG3 scenario, since the A3 and A4 sensors provide much higher local BFs compared to the others.

5 Conclusions

This study proposes a Bayesian hypothesis testing to detect subtle changes in modal properties caused by damage. A time series of actually observed accelerations of a bridge provides a likelihood function of the VAR model. The BF is formulated to detect anomalies in the extracted features. Bayesian hypothesis test for the BF is conducted. To investigate feasibility of the proposed approach for damage detection, this study examined a field experiment data on an actual steel truss bridge whose truss members were artificially severed. The VAR order is statistically determined using the Schwarz criterion.

The proposed method detected three different damage levels successfully. The global BF utilizing whole observations possibly indicates severity of damage. For the damage scenarios, where the truss members were totally cut, the measurement points close to the damaged members provide much higher local BFs than the others do. It even suggests possibility of damage localization for severely damaged members.

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