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Modified envelopes for seismic response vectors and its application to the reinforcement design of concrete structures subjected to stochastic ground motion

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In seismic analysis and design of structures, to evaluate the combined effects of response quantities simultaneously acting in a structural component, the statistical correlation between different seismic responses needs to be considered. The conventional response spectrum method provides the mean maximum values of individual responses, yet it offers no information on the statistical correlation between responses, and a direct combination of these maxima can be overly conservative for seismic design. To address the above problem, a response spectrum based procedure for predicting the envelope that bounds two or more responses in a linear structure is developed by Menun and Der Kiureghian (2000). The method is based on physical intuition, yet it lacks a rigorous theoretical basis. In this paper, a modified envelope method is proposed using concept of iso-density surface in the probability space of Gaussian random vectors. It is shown that compared with the original envelope method, the modified envelope approach can better capture the distributional shape of response vectors sampled from time history analysis. Finally, the modified envelope method is applied to the reinforcement design of concrete structures. Results of the proposed method are compared with results obtained from the original envelope method and the rectangular envelope method.

Keywords: Seismic analysis; Reinforcement design; Response spectrum; Random vibration

1 Introduction

In the reinforcement design of concrete structures, the reinforcement ratio of a structural component is determined by the response quantities (e.g. axial force, shear force, bending moment, etc.) acting in the component. At present, the reinforcement design of concrete structures subjected to earthquakes is typically performed based on the *rectangular envelope method*. The rectangular envelope method uses the mean maximum response values obtained from response spectrum analysis combined with static analysis to form a rectangular design domain, and the reinforcement ratio is computed using vertices of the rectangle. However, due to the difference in statistical correlations between response quantities, there is no guarantee that the response quantities of a structural component attain their maximum values at the same time. Therefore, the reinforcement ratio estimated using the rectangular envelope method can be overly conservative.

To address limitations of the conventional rectangular envelop method, Menun and Der Kiureghian proposed an *elliptical envelop method*. The method proposes an elliptical response envelope to replace the conventional rectangular envelope. The elliptical envelope is derived based on physical intuition, yet it lacks a rigorous theoretical basis. This paper provides a reinterpretation and modification of the elliptical envelope method.

The structure of this paper is as follows. In Section 2, response spectrum formulations used to develop the elliptical envelope method is introduced. Section 3 provides a review of the elliptical envelope method, and the modified elliptical envelope method is developed in Section 4. In Section 5, a numerical example of a frame structure is investigated to test and demonstrate the modified elliptical envelope method. Finally, concluding remarks are presented in Section 6.

2 Preliminaries

For a finite-element structural model, the internal forces on an element of the structure subjected to base acceleration can be expressed by

$$x_r(t) = q_r^T \boldsymbol{u}(t) = q_r^T \sum_{i=1}^{N} \phi_i \Upsilon_i s_i(t) \tag{1}$$

where q_r is the response transfer vector and it is a function of the element stiffness and the geometry of the structure, $\boldsymbol{u}(t)$ denotes the nodal displacements, ϕ_i is i-th modal vector, $\gamma_i = \phi_i^T MI / \phi_i^T M \phi_i$ is i-th modal participation factor, and $s_i(t)$ is the displacement response of an oscillator with the i-th modal frequency and damping. Let X_r denote the mean peak value of an individual response quantity in a time duration. If the structure is subjected to a uniform Gaussian ground motion excitation, X_r can be estimated by the CQC (Der Kiureghian 1981) combination rule

$$X_r^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} (q_i^T \phi_i \gamma_i)(q_j^T \phi_j \gamma_j) \rho_{ij} S_i S_j \tag{2}$$

where S_i is the mean peak displacement response of an oscillator that have the frequency and damping of the i-th mode, and S_i can be obtained from the displacement response spectrum, ρ_{ij} denotes the correlation coefficient between the responses of modes i and j, and it can be evaluated using the analytical formulation developed in (Der Kiureghian 1981; Wilson et.al. 1981).

Using a matrix formulation, Eq. (2) can be rewritten in a compact form as

$$X_r^2 = q_r^T Z q_r \tag{3}$$

$$Z = \boldsymbol{\Gamma}^T \boldsymbol{\Phi}^T \boldsymbol{S}^T \boldsymbol{R} \boldsymbol{S} \boldsymbol{\Phi} \boldsymbol{\Gamma} \tag{4}$$

where $\boldsymbol{S} = \text{diag}[S_i]$, $\boldsymbol{\Gamma} = \text{diag}[\gamma_i]$, and $\text{R} = \text{diag}[\rho_{ij}]$ are $\text{N} \times \text{N}$ diagonal matrices, and $\boldsymbol{\Phi} = [\phi_1, \phi_2, \dots, \phi_n]$ is the $\text{N} \times \text{N}$ modal matrix. Note that to compute different response quantities, one only needs to modify q_r, while matrix Z is constant.

In Eq. (3), X_r^2 represents the squared mean peak value of an individual response quantity $x_r(t)$. According to Menun and Der Kiureghian (2000), the cross term X_{rs} between peak responses of $x_r(t)$ and $x_s(t)$ can be expressed as

$$X_{rs} = q_r^T Z q_s \tag{5}$$

Consequently, all squared mean peak values of interest can be described in a "response matrix" written as

$$\text{X} = Q^T Z Q \tag{6}$$

where $\text{Q} = [q_1, q_2, \cdots, q_m]$ is a $\text{N} \times \text{m}$ matrix. The diagonal elements of X correspond to X_r^2, i.e. the squared mean peak values of various response quantities, while the off-diagonal elements of X correspond to X_{rs}, i.e. a quantity reflects the covariance between two responses. Note that Eq. (5) was proposed based on an analogy of Eq. (3), yet it lacks a rigorous theoretical basis. The development of the modified envelop method introduced in the following sections will disclose the flaw of Eq. (5).

3 Elliptical Envelope method

3.1 *Definition of the response boundary*

In the space of response vectors, a rectangular envelope can be determined by permutations of the positive and negative mean peak response values. A 2-dimensional response domain is shown in Fig. 1(a). Obviously, the rectangular response domain does not consider the correlation between response quantities. To address this problem, the elliptical envelope method replaces the rectangular response domain with an elliptical one. The specific procedure of the elliptical envelope method is described as follows.

In the response domain, define a unit direction α. The projection of $\mathbf{x}(\text{t}) = [x_1(t), x_2(t), \cdots, x_m(t)]^T$ on the unit direction in the m-dimensional response space can be expressed as,

$$x_\alpha(t) = \alpha^T \mathbf{x}(t) = Q^T \boldsymbol{u}(t) = q_\alpha^T \boldsymbol{u}(t) \tag{7}$$

where $q_\alpha^T = \alpha^T Q^T$. Comparing Eq. (1) with Eq. (7), it is easy to see that $X_\alpha = \max[x_\alpha(t)]$ can be estimated using the conventional response spectrum method. Similar to estimating the squared mean peak value of response quantity $x_r(t)$, the squared mean peak value of $x_\alpha(t)$ can be expressed as

$$X_\alpha^2 = q_\alpha^T Z q_\alpha = \alpha^T Q^T Z Q \alpha = \alpha^T X \alpha \tag{8}$$

Similar to the bounds in the direction of response axis, the boundary in direction α can be determined by X_α. Fig. 1(b) shows the bounds along direction $\pm$α in 2-dimensional response domain. Repeating the above process for all directions, the hyperplanes define a new response domain which is contained within the rectangular domain. Fig. 1(c) shows the new response domain in 2 dimensional response space. Clearly, the new response domain is an ellipse. Detailed derivation on the elliptical envelope can be found in Menun and Der Kiureghian (2000).

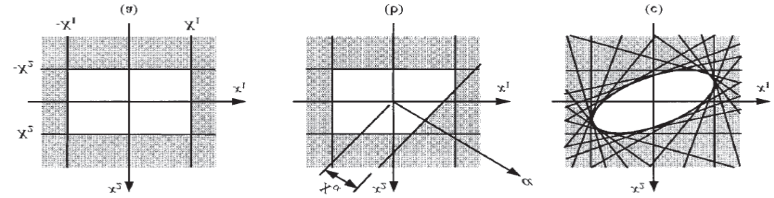

Figure 1. Construction of elliptical envelope (Menun and Der Kiureghian (2000))

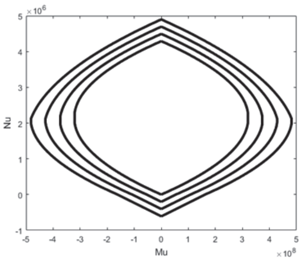

Figure 2. Capacity surface for different reinforcement ratios

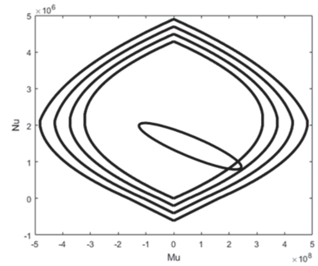

Figure 3. The optimal design of reinforcement ratio

3.2 *Method to determine the reinforcement ratio*

In the design of concrete buildings, to simplify the construction process, it is typical to design the cross sections of beams and columns as rectangular and the steels to be symmetrically distributed on two sides of the section. Therefore, the following analysis only considers the rectangular section and a symmetric reinforcement layout. Using the concrete structural design code of China, the capacity surface of structural components depends on the section size, concrete strength, reinforcement steel strength, and reinforcement steel ratio. Fig. 3 illustrates four capacity surfaces corresponding to a column with width and height $\mathrm{b} \times \mathrm{h} = 500\mathrm{mm} \times 500\mathrm{mm}$, concrete strength of $14.3N/mm^2$, reinforcement steel strength of $f_y = 360N/mm^2$, and different reinforcement ratios. It is seen from Fig. 2 that the capacity surfaces are approximately parallel with each other. In Fig.2, the capacity surface closer to the origin corresponds to a smaller reinforcement ratio. The domain inside the capacity surface corresponds to the safe domain, and the capacity surface is the limit-state surface between safe and failure domain. To guarantee structural safety, the response envelope should locate within the capacity surface. At the same time, to obtain an economical design, one should choose an optimal reinforcement ratio ρ_s^* such that the capacity surface is tangent to the response envelope. This idea is illustrated in Fig. 3. One may need a numerical algorithm to determine the optimal reinforcement ratio ρ_s^*.

4 Modified Elliptical Envelope Method

This section provides a reinterpretation and modification of the original elliptical envelope method using concepts of iso-density surfaces in the probability space of Gaussian vectors.

4.1 *Envelope for Gaussian random variables*

Consider a vector of n random variables $\mathrm{X} = [\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_\mathrm{n}]^\mathrm{T}$, let X follow multivariate Gaussian distribution with covariance matrix Σ_X. Without loss of generality, it is assumed X to be zero mean.

Now one is interested in obtaining an envelope in the space of X, so that samples generated from X are highly likely to fall into the envelope. Mathematically speaking, one needs to obtain a domain $\Omega \equiv \{\mathrm{X}|\mathrm{f}(\mathrm{X}) \leq 0\}$ such that the probability $\Pr[\mathrm{X} \in \Omega]$ is close to 1. In terms of the definition of Ω, the envelope is expressed by $\mathrm{f}(\mathrm{X}) = 0$. Although one has infinite choices for the forms of $\mathrm{f}(\mathrm{X}) = 0$, an obvious reasonable choice is a function with iso-density surface. Instead of finding $\mathrm{f}(\mathrm{X}) = 0$ in the space of X, we transform X into the standard normal space via

$$\mathrm{X} = \mathrm{Tu} \tag{9}$$

where u is a vector of uncorrelated standard normal random variables, and the transformation matrix T satisfies $\mathrm{TT}^\mathrm{T} = \Sigma_\mathrm{X}$. As long as X is multivariate Gaussian, the aforementioned transformation is one-on-one and exact. Due to the radical symmetry of the space of u, an iso-density envelope is hyper-spherical and can be expressed by

$$\mathrm{u}^\mathrm{T}\mathrm{u} - \mathrm{r}^2 = 0 \tag{10}$$

in which the radius r determines the probability that a random sample of u would fall into the hyper-spherical envelope. Transforming Eq. (10) back to the original space of X, one obtains

$$\mathrm{X}^\mathrm{T}{\Sigma_\mathrm{X}}^{-1}\mathrm{X} - \mathrm{r}^2 = 0 \tag{11}$$

Next, consider the case that $\mathrm{x_i}$ values outside the range $[-\mathrm{p_{Xi}}\sigma_{\mathrm{Xi}}, \mathrm{p_{Xi}}\sigma_{\mathrm{Xi}}]$, $\mathrm{i} = 1,2, \dots, \mathrm{n}$, where $\mathrm{p_{Xi}} > 0$ and σ_{Xi} is the standard deviation of the random variable $\mathrm{x_i}$, are of little practical importance. This indicates a cubic boundary is applied to the space of X. From a statistical point of view, a cubic boundary is not perfectly reasonable since the probability density along the boundary varies. Therefore, it is of interest to build an envelope with iso-density surface within confines of the cubic boundary. Similar to the aforementioned discussion, we transform the cubic boundary to the space of u. For the random variable x_i, the hyperplane boundary is defined by $x_i = \pm p_{Xi}\sigma_{Xi}$ in the space of X. Using Eq. (9), the transformed boundary in the space of u is $T_i u = \pm p_{Xi}\sigma_{Xi}$, where T_i is the i-th row vector of the transformation matrix T. The distance from the origin of u space to the transformed hyperplane boundary is

$$d_i = \frac{|\pm p_{Xi}\sigma_{Xi}|}{\sqrt{T_i T_i^T}} = \frac{p_{Xi}\sigma_{Xi}}{\sigma_{Xi}} = p_{Xi} \tag{12}$$

Eq. (12) implies that if one selects boundary for x_i as $[-p_{Xi}\sigma_{Xi}, p_{Xi}\sigma_{Xi}]$, $p_{Xi} = p$, $\forall i$, in the space of u one could build a hyper-spherical envelope that are tangent to all the transformed boundaries of $x_i = \pm p_{Xi}\sigma_{Xi}$. And if this hyper-spherical envelope is transformed back to the space of X (using Eq. (11) with $r = p$), a hyper-elliptical envelope tangent to all faces of the cubic boundary will be obtained. This idea is illustrated by Fig. 4. In Fig. 4, we set $n = 2$, $\Sigma_X = [1{,}0.5; 0.5{,}2]$, the boundary in the original space of X is set as $[-p_{Xi}\sigma_{Xi}, p_{Xi}\sigma_{Xi}]$, with $p_{Xi} = 3$, $i = 1{,}2$. To obtain Fig. 4, we first transform the rectangular boundary to the space of u, and then build a spherical envelope with radius $r = p_{Xi} = 3$. (According to Eq. (12), such a spherical envelope tangent to all the transformed bounds exists, and the radius equals to p.) Finally, the spherical envelope is transformed back to the space of X, and as expected the transformed envelope is tangent to all edges of the rectangular boundary. Note that in Fig. 4 a sample of 10000 random realizations of X are also shown in the space of X as well as in the space of u.On the other hand, if the hyperplane boundary $[-p_{Xi}\sigma_{Xi}, p_{Xi}\sigma_{Xi}]$ for x_i have $p_{Xi} \neq p_{Xj}$ for at least one pair of i, j, in the space of u a hyper-spherical envelope that are tangent to all the transformed boundaries does not exist. In this case a reasonable approach is to use a hyper-spherical envelope with radius $r = \max[p_{Xi} | i = 1,2,\dots,n]$ in the space of u, and in the space of X the corresponding hyper-elliptical envelope is only tangent to the $[-p_{Xi}\sigma_{Xi}, p_{Xi}\sigma_{Xi}]$ boundaries with largest p_{Xi}. This idea is illustrated by Fig. 5. In Fig. 5, we set $n = 2$, $\Sigma_X = [1{,}0.5; 0.5{,}2]$, the boundary in the original space of X is set as $[-p_{Xi}\sigma_{Xi}, p_{Xi}\sigma_{Xi}]$, with $p_{X1} = 2$, $p_{X2} = 3$. In Fig. 5 a sample of 10000 random realizations of X are also shown in the space of X as well as in the space of u.

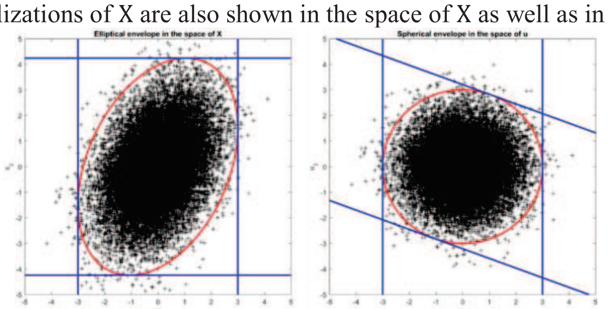

Figure 4. Envelopes in the space of X and u with identical p_i

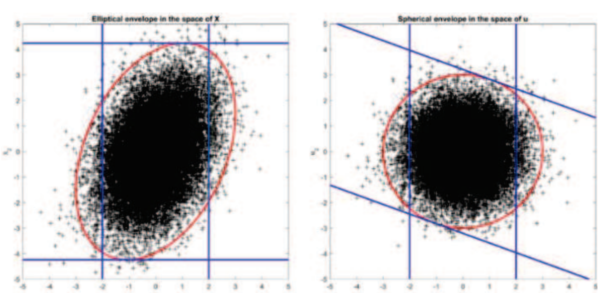

Figure 5. Envelopes in the space of X and u with various p_i

4.2 *Envelope for Gaussian processes*

Consider a vector of zero-mean stationary Gaussian processes $X(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$. For a specified duration T_d, the mean peak of $x_i(t)$ can be expressed as $E[\max x_i(t) \,|\, t \in T_d] = p_{Xi}\sigma_{Xi}$, where p_{Xi} is a peak factor and σ_{Xi} is the standard deviation of $x_i(t)$, and one is interested in $x_i(t)$ values inside the boundary $[-p_{Xi}\sigma_{Xi}, p_{Xi}\sigma_{Xi}]$. This indicates for an arbitrary time point t_k, the Gaussian random vector $X(t_k)$ has a cubic boundary. Now the situation is similar to the case studied in Section 4.1. In terms of the discussion in Section 4.1, a hyper-elliptical envelope tangent to all hyperplane boundaries $[-p_{Xi}\sigma_{Xi}, p_{Xi}\sigma_{Xi}]$, $i = 1,2,..,n$, can be obtained if $p_{Xi} = p$, $\forall i$, otherwise a reasonable approach is to obtain a hyper-elliptical envelope that is only tangent to the $[-p_{Xi}\sigma_{Xi}, p_{Xi}\sigma_{Xi}]$ boundary with largest p_{Xi}. Therefore, we can formulate a hyper-elliptical envelope expressed as

$$X^T{\Sigma_X}^{-1}X - {p_{X,max}}^2 = 0 \tag{13}$$

where Σ_X is the covariance matrix of $X(t)$, and $p_{X,max}$ is defined as

$$p_{X,max} = \max\left[p_{Xi} = \frac{E[\max x_i(t)|t \in T_d]}{\sigma_{Xi}} | i = 1,2,\dots,n\right] \tag{14}$$

4.3 ***Relation to Menun and Der Kiureghian's envelope approach***

To show the relationship between the proposed elliptical envelope and the one developed by Menun and Der Kiureghian, we write the covariance matrix Σ_X in Eq. (13) in terms of modal coordinates as

$$\Sigma_X = Q\Phi\Gamma\Lambda R\Lambda^T\Gamma^T\Phi^TQ^T \tag{15}$$

in which matrix Q represents the relationship between response quantities X and nodal displacements, $\Phi = [\phi_1, \phi_2, \dots, \phi_n]$ is the modal matrix, $\Gamma = \mathrm{diag}[\gamma_i]$ is a diagonal matrix of modal participate factors, $\Lambda = \mathrm{diag}[\sigma_i]$ is a diagonal matrix of standard deviations of modal responses, $R = [\rho_{ij}]$ is a correlation matrix between modal responses. Substitute Eq. (15) into Eq. (13), one has

$$X^T\left[Q\Phi\Gamma(p_{X,max}\Lambda)R(p_{X,max}\Lambda^T)\Gamma^T\Phi^TQ^T\right]^{-1}X - 1 = 0 \tag{16}$$

On the other hand, the envelope developed by Menun and Der Kiureghian is expressed as

$$X^T[Q\Phi\Gamma SRS^T\Gamma^T\Phi^TQ^T]^{-1}X - 1 = 0 \tag{17}$$

where $S = \mathrm{diag}[S_i]$ is a diagonal matrix of mean peak modal responses, and the other terms are the same as that in Eq. (16). Comparing Eq. (16) and Eq. (17), apparently the only difference between the two equations is the $p_{X,max}\Lambda$ and S term. Multiplying $p_{X,max}\Lambda$ with S^{-1}, and using $S_i = p_i\sigma_i$, where p_i is the modal peak factor, one can obtain

$$p_{X,max}\Sigma S^{-1} = \mathrm{diag}\left[\frac{p_{X,max}}{p_i}\right] \tag{18}$$

According to Eq. (18), if the modal peak factors p_i are only mildly different from each other and approximate $p_{X,max}$, it is expected that Eq. (16) and Eq. (17) would provide similar envelopes. This is usually the case if the modal spacing is not wide

Despite the similarities, the underlying philosophies in developing the two envelopes are different. The proposed approach is based on the concept of iso-density surfaces, so that the probability densities of $X(t)$ along the envelope are identical, while Menun and Der Kiureghian's approach is based on physical intuitions.

5 Numerical investigations

Consider a 9 story frame structure model. The floor height is 3m, the length of the span is 6m, the section size of the columns on the 1^{th}, 2^{th} and 3^{th} story is $750\text{mm} \times 750\text{mm}$, the section size of other columns is $700\text{mm} \times 700\text{mm}$, and the section size of all beams is $350\text{mm} \times 600\text{mm}$. The concrete strength is $f_c = 14.3N/mm^2$, and the reinforcing steel strength is $f_y = 360N/mm^2$. The reinforcement ratio of structural components are estimated using three envelope methods, i.e. the rectangular envelope method (REM), the elliptical envelope method (EEM) and the modified elliptical envelope method (MEEM). The ground motion is modeled by response spectrum specified from the seismic design code of China. The peak ground acceleration (PGA) of the ground motion is 0.12g. The power spectrum compatible with the response spectrum is obtained from the method described in Der Kiureghian and Neuenhofer 1992.

Fig. 6 a) and b) show the response envelopes for the column of the 4^{th} floor determined by the three methods. From the figures, it is clearly seen that the principal directions of the two elliptical envelopes are different. The original elliptical envelope method defines an envelope tangent to all faces of the rectangular boundary, while the modified method defines one only tangent to two faces of the boundary.

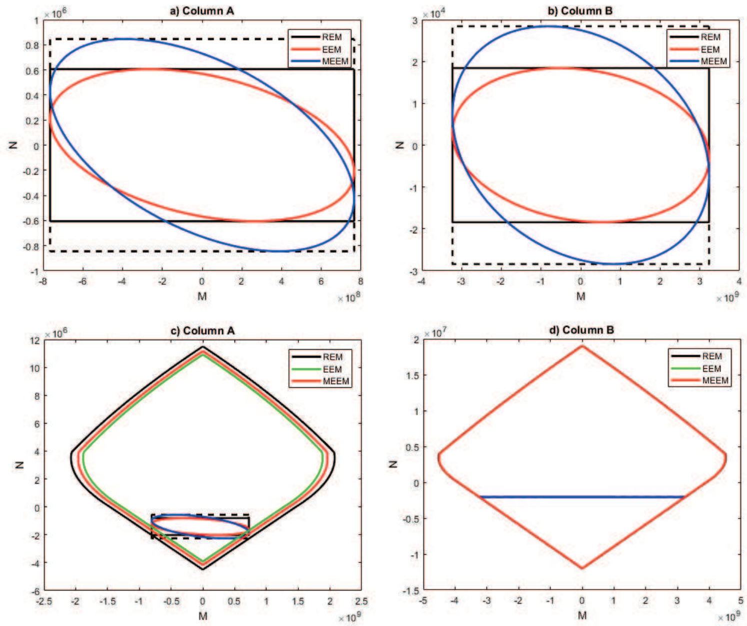

Figure 6. Envelopes and capacity from different method

Fig. 6 c) and d) show the capacity surfaces corresponding to the above three response envelopes. The results show that the reinforcement ratios determined using the EEM and the MEEM can be noticeably less than that of the rectangular envelope method. Thus the rectangular envelope method can be overly conservative. The reinforcement ratios of some components determined using the EEM is mildly less than that of the MEEM. Therefore it can be concluded that the EEM can lead to mildly unsafe designs for some structural components.

6 Conclusions

The paper proposes a reinterpretation and modification of the elliptical envelope method using concept of iso-density surfaces in the probability space of Gaussian vectors. The application of the modified elliptical method to the reinforcement design of concrete structures is illustrated using a 9 story frame structural model. The reinforcement ratios estimated from the modified elliptical envelope method is compared with results obtained from the original elliptical envelope method and the rectangular envelope method. It is found from the example that the original and modified elliptical envelope method can produce reinforcement ratios noticeably smaller than that obtained from the rectangular envelope method, suggesting the rectangular envelope method can be overly conservative. It is also found that the reinforcement ratios of some structural components determined using the original elliptical envelope method are mildly less than that obtained from the modified elliptical envelope method, suggesting that the original elliptical envelope method can lead to mildly unsafe designs for some structural components.

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