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A PDEM-BASED PERSPECTIVE TO ENGINEERING RELIABILITY: FROM STRUCTURAL ELEMENTS TO LIFELINE NETWORK

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Research of reliability of engineering structures has experienced a developing history for more than 90 years. However, the problem of how to resolve the global reliability of structural systems still remain open, especially the problem of the combinatorial explosion and the challenge of correlation between failure modes. Benefiting from the research of probability density evolution theory in recent years, the physics based system reliability researches open a new way for bypassing this dilemma. The present paper introduces the theoretical foundation of probability density evolution method in view of a broad background, whereby a probability density evolution equation for probability dissipative system is deduced. In conjunction of physical equations, a general engineering reliability analysis frame is then presented. For illustrative purposes, several cases are studied which proves the value of the proposed engineering reliability analysis method.

1 Introduction

Reliability based analysis and design of engineering structures and infrastructure systems underlies the safety of engineering systems. The pioneering investigations upon the uncertainty in the objective world and using the probability theory to deal with the engineering reliability assessment could be dated back to the early 20th century (Mayer, 1926; Rackwitz & Fiessler, 1978; Rackwitz, 2001). Owing to the outstanding contributions of Freudenthal (1947), Cornell (1969), Lind (1974) and Ang et al.(1975a), the engineering design paradigm based on the first-order second-moment (FOSM) method was built up during 1960's to 1970's. This paradigm was later developed rapidly and served as the foundations of worldwide national design provisions, which facilitated the establishment of the second-generation design theory of engineering structures (Li, 2016).

In fact, the crucial point of the second-generation design theory of engineering structures is to implement the approximate reliability analysis and design on the level of structural components based on the decomposition methodologies. This treatment consequently brings forward the basic contradictions inherent in the structural design theory (Li, 2016). In order to resolve these contradictions, great efforts have been made, especially on the researches on structural global reliability which can be traced back to the middle of 1960's. For example, in 1966, Freudenthal et al presented the upper bound of failure probability of series systems (Freudenthal et al., 1966). In 1975, Ang et al developed the probabilistic network method for the analysis of structural system reliability (Ang et al., 1975b). In 1979, Ditlevesen proposed the formulation of narrow limit estimation method for structural system reliability (Ditlevesen, 1979). Almost at the same time, Thoft-Christensen and Murotsu proposed the β-branch method and branch limit method based on joint probability, respectively (Thoft-Christensen & Murotsu, 1986). Although these methods prompted the wide formation of community consciousness on the research field of structural system reliability, the problem of combinatorial explosion and the challenge of failure probability correlation still remained open, resulting in the situation that the research of structural system reliability came into standstill since 1990's.

Actually, the problem of combinatorial explosion and failure probability correlation comes from the methodology research thought in which the main concern is focused on the failure results of structures other than the physical process of failure. During the first decade of 21th century, the research of probability density evolution theory gained eye-catching progresses, which brought a new dawn for the settlement of analysis and design of engineering system reliability. The thought of physical stochastic system research established the ideological basis for integrating the physical equation of engineering systems and the generalized probability density evolution equation, and pioneered a new way to implement the analysis and design of global reliability of structural systems in practical sense.

In the present paper, the theoretical foundation of probability density evolution method is first introduced in view of a broad background. Then a probability density evolution equation for probability dissipative system is deduced. In conjunction of the physical equation defined by a specific problem, a general engineering reliability analysis frame is presented. In order to show the feasibility of the proposed method, several case studies, including fatigue reliability of bridge structural elements, global reliability of high-rise buildings and functional reliability of water supply network, are discussed as examples.

2 Theoretical foundation of probability density evolution method

For a stochastic system, the principle of preservation of probability supplies a theoretical foundation for deriving the basic probability density evolution equation. This principle states that: if the random factors involved in a stochastic system are retained, then the probability will be preserved in the state evolution process of the system (Li & Chen, 2008; 2009).

In order to clarify the principle, we start with the investigation on a transform of a random function. Let ϖ be a basic random event and $X(\varpi)$ be a continuous variable with probability density function (PDF) $p_X(x)$, namely

$$\Pr\{X(\varpi)\in(x,x+\mathrm{d}x)\}=\mathrm{d}\Pr\{\varpi\}=p_X(x)\mathrm{d}x \tag{1}$$

where $\Pr\{\cdot\}$ is the probability measure.

Assume there exists a one to one mapping f from X to Y, that is

$$f: Y=f(X) \tag{2}$$

then the PDF of Y will be

$$p_Y(y)=p_X[f^{-1}(y)]\frac{\mathrm{d}x}{\mathrm{d}y} \tag{3}$$

Obviously, the above equation could be converted to

$$p_Y(y)\mathrm{d}y=p_X(x)\mathrm{d}x \tag{4}$$

Noticing that

$$\Pr\{Y(\varpi)\in(y,y+\mathrm{d}y)\}=\mathrm{d}\Pr\{\varpi\}=p_Y(y)\mathrm{d}y \tag{5}$$

it is evident that

$$\Pr\{Y(\varpi)\in(y,y+\mathrm{d}y)\}=\Pr\{X(\varpi)\in(x,x+\mathrm{d}x)\}=\mathrm{d}\Pr\{\varpi\} \tag{6}$$

This means that, in a mathematical transform, the probability measure will be preserved since the random events keep unchanged. This statement reveals the principle of preservation of probability. The principle is universally applicable to generic stochastic systems.

Noticing that a physical system can be described by a mathematical operator, without loss of generality, there exist

$$\mathscr{L}(\mathbf{Y},\partial^{(j)}\mathbf{Y},\mathbf{\Theta},x,t,\tau)=0 \tag{7}$$

where $\mathscr{L}(\cdot)$ denotes a general mathematical operator such as a differential operator or an integral operator; $\mathbf{Y}$ is a physical variable(s) which may be a vector, say in m dimensions, changing with spatial position and time; $\mathbf{\Theta}$ denotes a random vector which is actually a uncontrollable physical variables in the system; x and t are the spatial position variable and time variable, respectively; τ is an general time evolution parameter denoting the evolution direction of the system.

It is understood that, for a well-posed physical system described by Eq. (7), the solution $\mathrm{Y}(t)$ is existent, unique and continuously dependent on $\mathbf{\Theta}$. According to the principle of preservation of probability, it can be then deduced that the joint PDF of $(\mathbf{Y},\mathbf{\Theta})$ is governed by the following probability density evolution equation (PDEE) (Li & Chen, 2008; Li, 2009)

$$\frac{\partial p_{\mathbf{Y}\Theta}(\mathbf{y},\boldsymbol{\theta},\tau)}{\partial \tau}+\sum_{l=1}^{m}\dot{Y}_l(\boldsymbol{\theta},\tau)\frac{\partial p_{\mathbf{Y}\Theta}(\mathbf{y},\boldsymbol{\theta},\tau)}{\partial y_l}=0 \tag{8}$$

For a one-dimensional case, there exist

$$\frac{\partial p_{Y_l\Theta}(y_l,\boldsymbol{\theta},\tau)}{\partial \tau}+\dot{Y}_l(\boldsymbol{\theta},\tau)\frac{\partial p_{Y_l\Theta}(y_l,\boldsymbol{\theta},\tau)}{\partial y_l}=0 \tag{9}$$

This formulation provides a new understanding on the relationship between the physical world and the random world. Actually, if rewriting Eq. (9) as follows

$$\frac{\partial p_{Y_l\Theta}(y_l,\boldsymbol{\theta},\tau)}{\partial \tau}=-\dot{Y}_l(\boldsymbol{\theta},\tau)\frac{\partial p_{Y_l\Theta}(y_l,\boldsymbol{\theta},\tau)}{\partial y_l} \tag{10}$$

we could realize such an important fact immediately: the evolution of probability density of a stochastic relies on the change of physical state of the system! It demonstrates in an elegant manner that the evolution of probability density obeys a rigorous physical law instead of being rule less. Obviously, this understanding comes up with a new perspective about the real world.

3 PDEE for Probability Dissipative Systems

For a stochastic system, the probability dissipation could take place at any time in evolutionary process. Concerning the first passage problems, for example, when the response of the system crosses a specified level, the adherent probability of the path will be dissipated, which results in a probability-dissipated system. Another example is the structural dynamic stability. Once the stability criterion is violated, the corresponding probability of the path will be dissipated, which also results in a probability-dissipated system. Obviously, the probability density evolution equation that governs such probability-dissipated systems is important for obtaining the response of such systems.

Without loss of generality, we assume that the probability dissipation take place in the time interval $[t,t+\Delta t]$, the dissipated probability can be then denoted as $-\mathscr{H}\left(\mathbf{Y}\left(t\right)\right)P_r\left(\mathbf{Y}\left(t\right)\right)$. Here

$$\mathscr{H}\left(\mathbf{Y}\left(t\right)\right)=\begin{cases}0 & \mathbf{Y}\left(t\right)\notin\Omega_D\\ 1 & \mathbf{Y}\left(t\right)\in\Omega_D\end{cases} \tag{11}$$

Ω_D is the probability dissipation domain; $\mathscr{H}$ is the identity indicator of probability dissipation. It is indicated that when $\mathbf{Y}\left(t\right)$ reaches a critical state resulting in that the physical quantity of interest enters into the domain Ω_D, the value of $\mathscr{H}$ turns to be one from the time instant t. Therefore, the symbol $\mathscr{H}$ may be called as the screening operator.

Using the description of probability density, the dissipated probability can be then expressed as

$$\delta P=-\mathscr{H}\left(\mathbf{Y}\left(t\right)\right)P_r\left(\mathbf{Y}\left(t\right)\right)=-\mathscr{H}\left(\mathbf{Y}\left(t\right)\right)\left[\int_{-\infty}^{+\infty}p_{\mathrm{Y}}\left(\mathbf{y},t\right)\mathrm{d}\mathbf{y}\right] \tag{12}$$

where δP is the dissipated probability.

It is noted that all the randomness involved in $\mathbf{Y}\left(t\right)$ comes from $\mathbf{\Theta}$, and the joint PDF of augmented system $\left(\mathbf{Y}\left(t\right),\mathbf{\Theta}\right)$ can be represented as $p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)$. Alternatively, the dissipated probability in the form of the joint PDF $p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)$ is

$$\delta P=-\mathscr{H}\left(\mathbf{Y}\left(\boldsymbol{\theta},t\right)\right)\left[\int_{\Omega_t\times\Omega_{\mathbf{\Theta}}}p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)\mathrm{d}\mathbf{y}\mathrm{d}\boldsymbol{\theta}\right] \tag{13}$$

where Ω_t and $\Omega_{\mathbf{\Theta}}$ are the distribution domains of $\mathbf{Y}$ at time instant t and of $\mathbf{\Theta}$, respectively.

If the dissipated probability at time instant t is taken as the average probability dissipated during the time interval $\left[t,t+\Delta t\right]$, we can define the average joint PDF as

$$\overline{p}_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)=\frac{p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)}{\Delta t} \tag{14}$$

Obviously, when $\Delta t \to 0$, the following relationship exists

$$\overline{p}_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)=p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right) \tag{15}$$

which reveals that the average joint PDF dissipated during the interval Δt is actually the joint PDF at the time instant t when Δt approaches to zero.

Then, the dissipated probability could be rewritten as

$$\delta P=-\mathscr{H}\left(\mathbf{Y}\left(\boldsymbol{\theta},t\right)\right)\cdot\left[\int_{\Omega_t\times\Omega_{\boldsymbol{\Theta}}}\overline{p}_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)\Delta t\mathrm{d}\mathbf{z}\mathrm{d}\boldsymbol{\theta}\right] \tag{16}$$

Let

$$\tilde{\mathbf{y}}=\mathbf{y}\left(t+\Delta t\right) \tag{17}$$

Then the following equation holds for a probability dissipated system in the time interval $[t, t+\Delta t]$ according to the principle of preservation of probability

$$p_{\mathbf{Y\Theta}}\left(\tilde{\mathbf{y}},\boldsymbol{\theta},t+\Delta t\right)\mathrm{d}\tilde{\mathbf{y}}=p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)\mathrm{d}\mathbf{y}-\mathscr{H}\left(\mathbf{Y}\left(\boldsymbol{\theta},t\right)\right)\overline{p}_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)\Delta t\mathrm{d}\mathbf{y} \tag{18}$$

where

$$\mathrm{d}\tilde{\mathbf{y}}=|J|\,\mathrm{d}\mathbf{y} \tag{19}$$

Here J is the Jacobian and it can be verified that $\left|J\right|=1$ (Xu, 2014) .

Noticing that

$$\begin{aligned}&p_{\mathbf{Y\Theta}}\left(\mathbf{y}\left(t+\Delta t\right),\boldsymbol{\theta},t+\Delta t\right)\\&=p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)+\left[\dot{\mathbf{y}}\left(\boldsymbol{\theta},t\right)\frac{\partial p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)}{\partial\mathbf{y}}+\frac{\partial p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)}{\partial t}\right]\Delta t\\&=p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)+\left[\frac{\partial p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)}{\partial t}+\sum_{l=1}^{m}\dot{\mathrm{y}}_l\left(\boldsymbol{\theta},t\right)\frac{\partial p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)}{\partial y_l}\right]\Delta t\end{aligned} \tag{20}$$

Substituting Eq. (19) and (20) into Eq. (18) will yields

$$\frac{\partial p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)}{\partial t}+\sum_{l=1}^{m}\dot{\mathrm{y}}_l\left(\boldsymbol{\theta},t\right)\frac{\partial p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right)}{\partial \mathrm{y}_l}=-\mathscr{H}\left(\mathbf{Y}\left(\boldsymbol{\theta},t\right)\right)p_{\mathbf{Y\Theta}}\left(\mathbf{y},\boldsymbol{\theta},t\right) \tag{21}$$

where the relationship in Eq.(15) is introduced.

Eq. (21) is called the generalized probability density evolution equation for probability dissipated system (GDEE-PD). Specifically, when $m=1$, the Eq. (21) reduces to be

$$\frac{\partial p_{Y_l\boldsymbol{\Theta}}\left(y_l,\boldsymbol{\theta},t\right)}{\partial t}+\dot{\mathrm{Y}}_l\left(\boldsymbol{\theta},t\right)\frac{\partial p_{Y_l\boldsymbol{\Theta}}\left(y_l,\boldsymbol{\theta},t\right)}{\partial y_l}=-\mathscr{H}\left(\mathbf{Y}\left(\boldsymbol{\theta},t\right)\right)p_{Y_l\boldsymbol{\Theta}}\left(y_l,\boldsymbol{\theta},t\right) \tag{22}$$

which can be called as the one-dimensional generalized density evolution equation for probability dissipated system.

Under the initial condition,

$$p_{Y\boldsymbol{\Theta}}\left(y,\boldsymbol{\theta},t\right)\Big|_{t=t_0}=\delta\left(y-y_0\right)p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right) \tag{23}$$

the partial differential equation Eq. (22) can be solved cooperating with the physical equation such as Eq. (7). In Eq. (23), y_0 is the deterministic initial condition.

When the system variable $\mathbf{Y}(t)$ is considered in different domain, Eq. (22) has different solution, there exist (i). When $\mathbf{Y}\left(\boldsymbol{\theta},t\right)\notin\Omega_D$, $\mathscr{H}\left(\mathbf{Y}\left(\boldsymbol{\theta},t\right)\right)=0$, Eq. (22) then turns to be the generalized density evolution equation for probability preserved system

$$\frac{\partial p_{Y_l\boldsymbol{\Theta}}\left(y_l,\boldsymbol{\theta},t\right)}{\partial t}+\dot{Y}_l\left(\boldsymbol{\theta},t\right)\frac{\partial p_{Y_l\boldsymbol{\Theta}}\left(y_l,\boldsymbol{\theta},t\right)}{\partial y_l}=0 \tag{24}$$

where the non-zero solution $p_{Y_l\boldsymbol{\Theta}}\left(y_l,\boldsymbol{\theta},t\right)$ can be obtained.

(2). When $\mathbf{Y}\left(\boldsymbol{\theta},t\right)\in\Omega_D$, indicating that $\mathbf{Y}\left(\boldsymbol{\theta},t\right)$ arrives at its criticality, $\mathscr{H}\left(\mathbf{Y}\left(\boldsymbol{\theta},t\right)\right)=1$, Eq.(22) becomes

$$\frac{\partial p_{Y_l\boldsymbol{\Theta}}\left(y_l,\boldsymbol{\theta},t\right)}{\partial t}+\dot{Y}_l\left(\boldsymbol{\theta},t\right)\frac{\partial p_{Y_l\boldsymbol{\Theta}}\left(y_l,\boldsymbol{\theta},t\right)}{\partial y_l}=-p_{Y_l\boldsymbol{\Theta}}\left(y_l,\boldsymbol{\theta},t\right) \tag{25}$$

which indicates that increments of the joint PDF is a negative joint PDF, on the other words, it means probability dissipation. Therefore, Eq. (25) has a zero solution, i.e. $p_{Y_l\boldsymbol{\Theta}}\left(y_l,\boldsymbol{\theta},t\right)=0$.

4 Structural System Reliability Analysis

The above-mentioned theoretical background actually supplies a broad possibility to solve engineering reliability problem. In fact, the screening operator $\mathscr{H}$ may be defined in a more general form as follows

$$\mathscr{H}\left[f\left(\mathbf{Y}\left(\boldsymbol{\theta},t\right)\right)\right]=\begin{cases}0 & f\left(\mathbf{Y}\left(\boldsymbol{\theta},t\right)\right)\in\Omega_S\\ 1 & f\left(\mathbf{Y}\left(\boldsymbol{\theta},t\right)\right)\in\Omega_D\end{cases} \tag{26}$$

where $\mathbf{Y}\left(\boldsymbol{\theta},t\right)$ is structural response ; $f\left(\cdot\right)$ is a general function which relies upon the specific failure criteria of structures; Ω_S is the safety domain of structures and Ω_D is the failure domain of structures. Obviously, $\Omega_S\cap\Omega_D=\phi$.

On the other hand, for general engineering systems, when the mechanical behavior is concerned, Eq. (7) can be expressed as a set of solid mechanics equation as follows

$$\begin{cases}\nabla\boldsymbol{\sigma}+\mathbf{b}=\rho\ddot{\mathbf{u}}+\eta\dot{\mathbf{u}}\\ \boldsymbol{\varepsilon}=\dfrac{1}{2}\left(\nabla\mathbf{u}+\nabla^T\mathbf{u}\right)\\ \dot{\boldsymbol{\sigma}}=G(\dot{\boldsymbol{\varepsilon}})\end{cases} \tag{27}$$

where ∇ is the partial differential operator, $\boldsymbol{\sigma}$ is the stress tensor, $\mathbf{b}$ is the body force, ρ is the density of material, η is the viscous damping coefficient, $\boldsymbol{\varepsilon}$ is the strain tensor, $\mathbf{u}$ is the displacement vector, the over dot denotes differentiation in terms of time. $G(\cdot)$ denotes a general function or operator.

Then, taking a representative physical quantity (for example, the displacement of a specified element of the structure, $U_p(t)$) as the observed variable, a set of structural reliability equations could be established:

$$\begin{cases}\nabla\boldsymbol{\sigma}+\mathbf{b}=\rho\ddot{\mathbf{u}}+\eta\dot{\mathbf{u}}\\ \boldsymbol{\varepsilon}=\dfrac{1}{2}\left(\nabla\mathbf{u}+\nabla^T\mathbf{u}\right)\\ \dot{\boldsymbol{\sigma}}=G(\dot{\boldsymbol{\varepsilon}})\\ \dfrac{\partial p_{U_p\boldsymbol{\Theta}}\left(u_p,\boldsymbol{\theta},t\right)}{\partial t}+\dot{U}_p\left(\boldsymbol{\theta},t\right)\dfrac{\partial p_{U_p\boldsymbol{\Theta}}\left(u_p,\boldsymbol{\theta},t\right)}{\partial u_p}=-\mathscr{H}\left[f(\mathbf{u}(\boldsymbol{\theta},t))\right]p_{U_p\boldsymbol{\Theta}}\left(u_p,\boldsymbol{\theta},t\right)\\ p_{U_p\boldsymbol{\Theta}}\left(u_p,\boldsymbol{\theta},t\right)\Big|_{t=t_0}=\delta\left(u_p-u_{p0}\right)p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)\end{cases} \tag{28}$$

Solving these equations will derive the joint PDF $p_{U_p\boldsymbol{\Theta}}\left(u_p,\boldsymbol{\theta},t\right)$ and the PDF of $U_p(t)$ can then be obtained by

$$p_{U_p}\left(u_p,t\right)=\int_{\Omega_{\boldsymbol{\Theta}}} p_{U_p\boldsymbol{\Theta}}\left(u_p,\boldsymbol{\theta},t\right)\mathrm{d}\boldsymbol{\theta} \tag{29}$$

While the reliability of the structure corresponding to the specified failure criteria will be

$$\mathrm{R(t)}=\int_{\Omega_{u_p}} p_{U_p}\left(u_p,t\right)\mathrm{d}u_p \tag{30}$$

In order to verify the applicability of the proposed method, several typical cases will be studied in the following sections.

4.1 *Fatigue Reliability of Bridge Structures*

Most structures are subjected to cyclic loads during their service life, such as bridge decks, wind-turbine blades, pavements of high way and airport etc. In many circumstances, fatigue or time-delayed damage will occur in these structures. A number of investigations show that, especially for bridge structures, the fatigue life has a significant variation. Therefore, the fatigue life prediction is very important for such kind of structures.

From a viewpoint of multiscale physical mechanics, a stochastic damage constitutive model has been developed (Ding & Li, 2017), in which the basic constitutive equation of concrete materials is given by

$$\boldsymbol{\sigma}=(\mathbf{I}-D^{+}\mathbf{P}^{+}-D^{-}\mathbf{P}^{-}):\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathbf{p}}) \tag{31}$$

where $\mathbf{I}$ is a fourth order unit tensor, D^{+},D^{-} are the tensile and compressive damage variables, respectively, $\mathbf{P}^{+},\mathbf{P}^{-}$ are the fourth positive and negative projection tensor, $\boldsymbol{\varepsilon}^{p}$ is the plastic stress tensor, $\mathbf{C}$ is the fourth tensor of initial modulus of elasticity.

The damage variable is defined as

$$\begin{cases} D^{\pm}=\int_0^1\left[1-H\left(E_s-E_f\right)H(E_s-Y)\right]\mathrm{d}x \\ E_f=\int_0^t C_0 e^{-\kappa d}Y(Y-\gamma(\vartheta))\left(\dfrac{Y}{\Gamma e^{-\beta\vartheta}}\right)^p \mathrm{d}t \\ E_s=\dfrac{1}{2}E_0\Delta^2(x) \end{cases} \tag{32}$$

where $H(\cdot)$ is Heavide's function, E_s denotes the inherent energy of the representative volume element, E_f is the total energy dissipation at mesoscale, κ is a material constant, Y denotes the damage energy release rate conjugated with damage variable in function of Helmholtz free energy, ϑ is an equivalent accumulation strain, β,C_0 are constant coefficients, $\gamma(\cdot)$ is the modified surface energy, Γ is the representative volumetric homogenized surface energy, p is a critical exponent, E_0 denotes the initial elastic modulus and Δx denotes the denotes one-dimensional random fracture strain field.

Integrating the constitutive equation with equilibrium equation and geometric equation will give the basic physical equation such as Eq. (7). While the screening operator $\mathscr{H}$ may be defined in a damage criteria form as following

$$\mathscr{H}\left[D_{\max}(\boldsymbol{\theta},t)\right]=\begin{cases}0 & D_{\max}(\boldsymbol{\theta},t)\le[d]\\ 1 & D_{\max}(\boldsymbol{\theta},t)>[d]\end{cases} \tag{33}$$

where $[d]$ is the permitted damage.

Then the basic fatigue reliability equation of concrete structures is summarized as follows

$$\begin{cases} \nabla\boldsymbol{\sigma} + \mathbf{b} = \rho\ddot{\mathbf{u}} + \eta\dot{\mathbf{u}} \\ \boldsymbol{\varepsilon} = \dfrac{1}{2}\left(\nabla\mathbf{u} + \nabla^T\mathbf{u}\right) \\ \boldsymbol{\sigma} = (\mathbf{I} - D^{+}\mathbf{P}^{+} - D^{-}\mathbf{P}^{-}) : \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}) \\ \dfrac{\partial p_{U_p\boldsymbol{\Theta}}\left(u_p, \boldsymbol{\theta}, t\right)}{\partial t} + \dot{U}_p\left(\boldsymbol{\theta}, t\right)\dfrac{\partial p_{U_p\boldsymbol{\Theta}}\left(u_p, \boldsymbol{\theta}, t\right)}{\partial u_p} = -\mathscr{H}\left[D_{\max}(\mathbf{u}, \boldsymbol{\theta}, t)\right] p_{U_p\boldsymbol{\Theta}}\left(u_p, \boldsymbol{\theta}, t\right) \\ p_{U_p\boldsymbol{\Theta}}\left(u_p, \boldsymbol{\theta}, t\right)\Big|_{t=t_0} = \delta\left(u_p - u_{p0}\right) p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right) \end{cases} \tag{34}$$

After obtaining the joint PDF $p_{U_p\boldsymbol{\Theta}}\left(u_p, \boldsymbol{\theta}, t\right)$, the fatigue reliability of the structures can be derived by Eqs. (27) and (28).

For illustrative purposes, the fatigue life analyses of high-speed and heavy-load railway bridges are addressed as numerical examples. In China, the axial load of train increases from 23 tons to 30 tons after the ordinary railway changed to the heavy-load train. For this reason, the fatigue performance of existing prestressed concrete bridges distributed along the railway needs to be evaluated. Here the simply-supported beams commonly used in the railway lines are addressed, of which the fatigue damage evolution of the concrete in the compression zone is investigated. The average damage of the concrete in the compression zone of the beam is taken as basic damage variable and the threshold of fatigue damage is taken as the damage level corresponding to the compressive residual strain which is 0.4 times of the axial compression strength. Using the above-mentioned principles, the fatigue reliability and service life prediction of the heavy-load railway bridge element are carried out. Some results are shown in Figures 1 and 2.

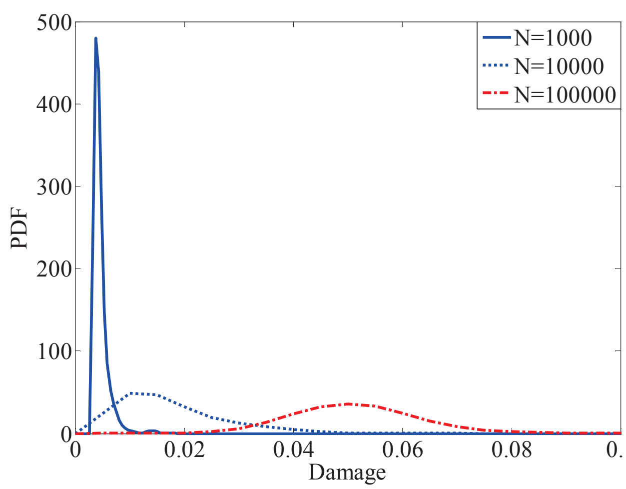


Figure 1 Probability density of fatigue damage under different cycles of loadings.

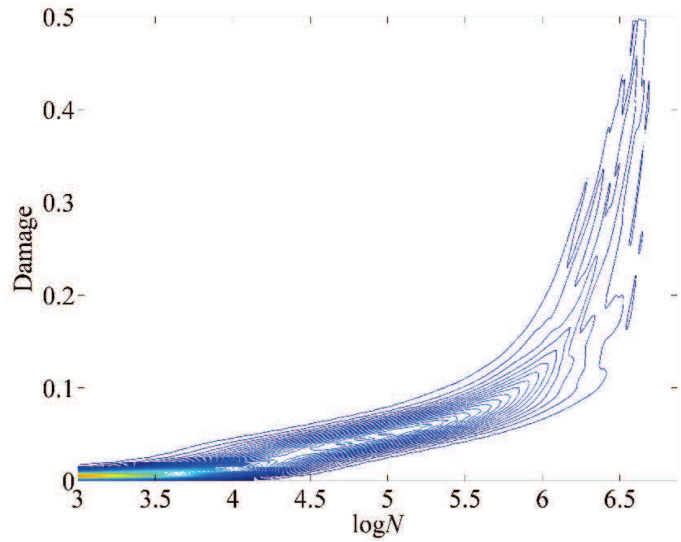


Figure 2 Contour of probability density of fatigue damage.

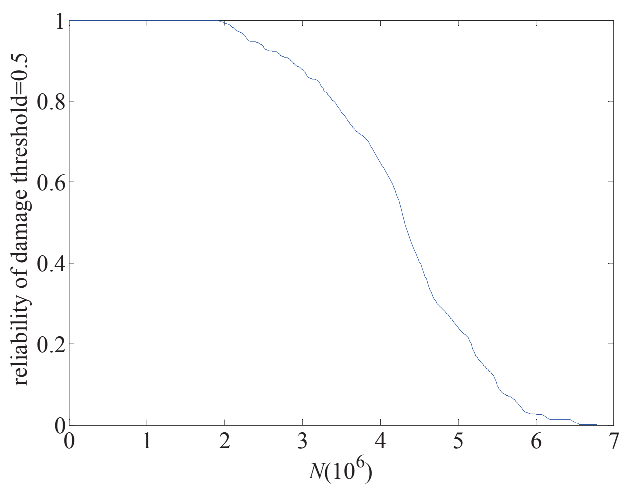


Figure 3 Fatigue reliability of the concrete beam.

The quantitative results of fatigue reliability are listed in Tables 1 and 2. It is seen that integrated with physical equation and GDEE-PD, the life-cycle fatigue analysis of structures could implement the elegant assessment of fatigue reliability and accurate prediction of life-cycle period.

Table 1 Fatigue reliability of different fatigue life

Fatigue cycles	2000000	2500000	3000000	3500000	4000000	4500000	5000000
Fatigue reliability	0.9932	0.9342	0.8788	0.7744	0.6500	0.4057	0.2405

Table 2 Fatigue life of different fatigue reliability

Fatigue reliability	0.99	0.95	0.90	0.80	0.70	0.60	0.50
Fatigue cycles	2,046,644	2,301,441	2,857,590	3,400,167	3,845,917	4,139,996	4,315,190

4.2 ***Global reliability of high-rise buildings***

As discussed in the section 'Introduction', global reliability assessment of structures has been a challenging issue in the past 40 years. Taking high-rise buildings as studied object, a series of researches were carried out in recent years. In the research, an energy-based structural collapse criterion is proposed for the collapse assessment of structures. Meanwhile, to address the uncertainty propagation in a complex nonlinear dynamic system, the PDEM is adopted as a feasible solution (Li, Zhou & Ding, 2017).

In fact, for general structures, there are different levels of structural failure criteria, each of them correspond its own screening operator $\mathscr{H}$. For example, for the beam or column failure of structure, $\mathscr{H}$ can be defined in a moment criteria form as follows

$$\mathscr{H}\left[M_{\max}(\boldsymbol{\theta},t)\right]=\begin{cases}0 & M_{\max}(\boldsymbol{\theta},t)\le[M_u]\\ 1 & M_{\max}(\boldsymbol{\theta},t)>[M_u]\end{cases} \tag{35}$$

where $M_{\max}$ is the maximum moment in a beam or column; M_u is the limit moment of a beam or column.

In this level, structural reliability assessment is focused on the structural elements. Appling for the equivalent extreme-value principle (Li, Chen & Fan, 2007), the global reliability can be evaluated. On the other hand, when considering the seismic collapse probability analysis for large complex reinforced concrete structures, an energy-based structural collapse criterion may be introduced, and $\mathscr{H}$ can be defined based on an energy-based structural collapse criterion as follows

$$\mathscr{H}\left[S(\mathbf{u},\boldsymbol{\theta},t)\right]=\begin{cases}0 & S(\mathbf{u},\boldsymbol{\theta},t)\le 0\\ 1 & S(\mathbf{u},\boldsymbol{\theta},t)>0\end{cases} \tag{36}$$

where

$$S(u,t)=E_{\text{eff\_inp}}(u,t)-E_{\text{eff\_intr}}(u,t) \tag{37}$$

$$E_{\text{eff\_inp}}(u,t)=\int_0^t\mathbf{F}^{\mathrm{T}}(t)\mathrm{d}\mathbf{u}(t)-\int_0^t\dot{\mathbf{u}}^{\mathrm{T}}(t)\mathbf{C}\dot{\mathbf{u}}(t)\mathrm{d}t-\int_0^t(\int_V\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}_{\mathrm{p}}\,\mathrm{d}V)\mathrm{d}t \tag{38}$$

$$E_{\text{eff\_intr}}(u,t)=\left|\mathbf{f}^{\mathrm{T}}(\mathbf{u},t)\mathbf{u}(t)-\int_V\boldsymbol{\sigma}:\dot{\boldsymbol{\varepsilon}}_{\mathrm{e}}\,\mathrm{d}V\right| \tag{39}$$

where $E_{\text{eff\_inp}}(u,t)$ is the effective external work to the system at any time t ; $\boldsymbol{\sigma}$ is the stress tensor; $\dot{\boldsymbol{\varepsilon}}_{\mathrm{e}}$ is the elastic strain rate tensor; V denotes the solution domain; $E_{\text{eff\_intr}}(u,t)$ denotes the absorbing energy that belongs to the structural system induced by the vibration of the system at time t; $\dot{\boldsymbol{\varepsilon}}_{\mathrm{p}}$ is the plastic strain rate tensor.

Then by employing the above analytical principle, the basic governing equation for analyzing the global reliability will be as follows

$$\begin{cases}\nabla\boldsymbol{\sigma}+\mathbf{b}=\rho\ddot{\mathbf{u}}+\eta\dot{\mathbf{u}}\\ \boldsymbol{\varepsilon}=\dfrac{1}{2}\left(\nabla\mathbf{u}+\nabla^T\mathbf{u}\right)\\ \boldsymbol{\sigma}=(\mathbf{I}-D^+\mathbf{P}^+-D^-\mathbf{P}^-):\mathbf{C}:(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^{\mathbf{p}})\\ \dfrac{\partial p_{U_p\boldsymbol{\Theta}}\left(u_p,\boldsymbol{\theta},t\right)}{\partial t}+\dot{U}_p\left(\boldsymbol{\theta},t\right)\dfrac{\partial p_{U_p\boldsymbol{\Theta}}\left(u_p,\boldsymbol{\theta},t\right)}{\partial u_p}=-\mathscr{H}\left[S(\mathbf{u},\boldsymbol{\theta},t)\right]p_{U_p\boldsymbol{\Theta}}\left(u_p,\boldsymbol{\theta},t\right)\\ p_{U_p\boldsymbol{\Theta}}\left(u_p,\boldsymbol{\theta},t\right)\Big|_{t=t_0}=\delta\left(u_p-u_{p0}\right)p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)\end{cases} \tag{40}$$

Obviously, after solve above equations, the global reliability of structures can be derived by Eqs. (27) and (28).

For illustrative purposes, an 18-storey high-rise RC frame-shear wall building which is located in Shanghai is taken as an example. The finite element model of the structure is shown in Figure 4, and totally 53,372 elements are involved. The stochastic dynamic analysis herein is only in terms of the random seismic input while taking no account of the uncertainty from structural properties. In this regard, the mean values of all the material properties are adopted in the analysis.

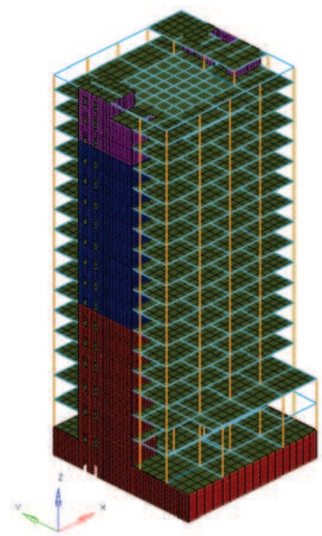
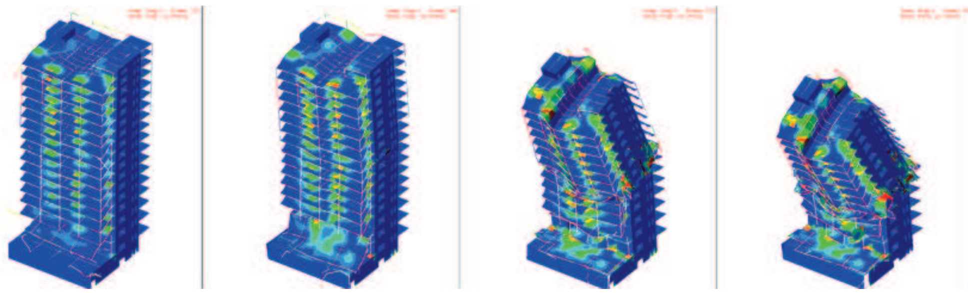


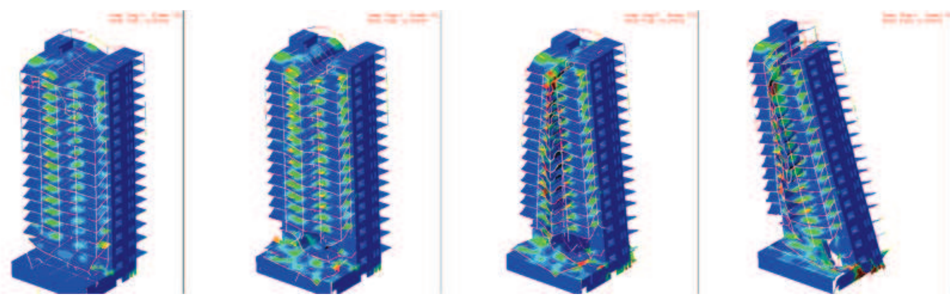
Figure 4 The finite element model of an 18-storey building.

Based on the numerical platform developed for stochastic analysis of structures, the nonlinear seismic responses of the structure under stochastic ground motions are attained manifesting with quite different failure paths and patterns. Two typical structural collapse processes and modes are depicted in Figure 5. It can be seen that, because of coupling effect of developing process of nonlinearity and stochastic input, the initial damage locations and occurrence time as well as the subsequent damage evolutions of the structure will be a typical random process, and therefore providing different structural collapse modes.



t=21.8 s t=24.9 s t=28.3 s t=29.7 s

(a) Sample 1



t=13.4 s t=21.6 s t=28.1 s t=32.6 s

(d) Sample 2

Figure 5 Typical collapse processes and failure modes of the high-rise building.

Figure 6 shows three typical PDFs of the inter-story drift ratio (ISDR) at certain instants of time. It is seen that the PDFs are quite different from those widely used regular probability distributions. These results indicate that the structural response process is a complex stochastic damage evolution process and should be investigated from the development process of nonlinearity.

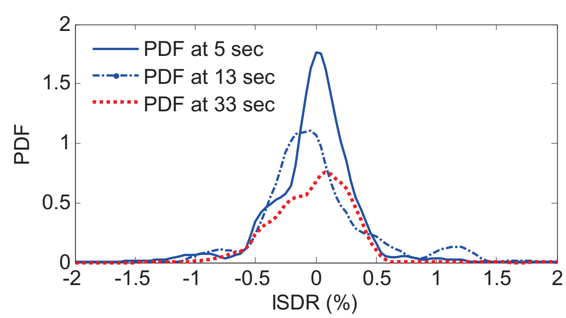


Figure 6 Typical PDFs of the ISDR responses at certain instants of time.

The global reliabilities against collapse of the structure are pictured in Figure 7. It can be seen that the reliability is changing with time along the seismic process.

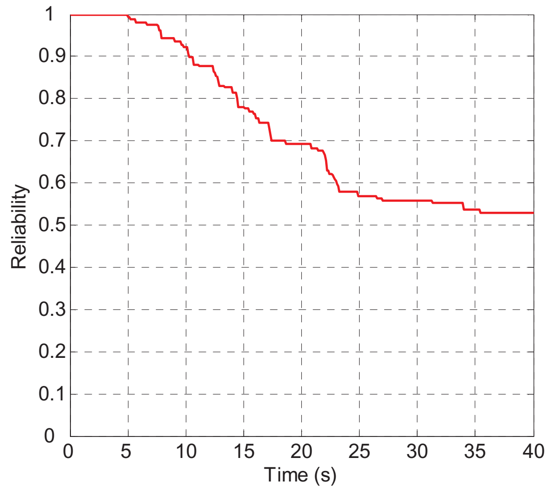


Figure 7 Global reliability of structures by energy criterion.

4.3 Functional reliability of water supply network

The water supply network is a kind of infrastructure system for modern cities, which include buried pipes, pumps, and valves, etc., to deliver water from sources to customer. Many previous earthquake investigations showed that the seismic performance of water supply network is very fragile. Applying above principle to the serviceability analysis of water supply network under earthquakes, the service reliability (also known as functional reliability) under and after an earthquake could be derived quantitatively.

For a water supply system, the basic physical equation is the transient flow analysis equations which are constructed by a momentum equation and a continuity equation

$$\frac{1}{gA}\left(\frac{\partial Q(t)}{\partial t}+V(t)\frac{\partial Q(t)}{\partial x}\right)+\frac{\partial H(t)}{\partial x}+fQ(t)\left|Q(t)\right|^{1-m}=0 \tag{41}$$

$$\frac{\partial H(t)}{\partial t}+V(t)\frac{\partial H(t)}{\partial x}+\frac{a^2}{gA}\frac{\partial Q(t)}{\partial x}=0 \tag{42}$$

where g is the acceleration of gravity; A is the cross sectional area of the pipe; Q is the flow rate in pipeline; V is the fluid velocity; H is the pressure head; f and m are two coefficients of friction resistance, which depend on different hydraulic loss models; a is the propagation velocity of small disturbances in a pipe.

Using the characteristic line method, the flow rate in pipeline and the flow pressure of each node can be derived from the above differential equations. However, since the seismic ground motion is a stochastic process, the pipeline damage and leakage after an earthquake are both random events. Taking the critical random variables associated with ground motions and pipe systems as Θ, and taking flow pressure at each node of water supply network as an analytical variable, the functional reliability of water supply network can be then derived by solving the following equations

$$
\begin{cases}
\dfrac{1}{gA}\left(\dfrac{\partial Q(t)}{\partial t}+V(t)\dfrac{\partial Q(t)}{\partial x}\right)+\dfrac{\partial H(t)}{\partial x}+fQ(t)\left|Q(t)\right|^{1-m}=0\\
\dfrac{\partial H(t)}{\partial t}+V(t)\dfrac{\partial H(t)}{\partial x}+\dfrac{a^2}{gA}\dfrac{\partial Q(t)}{\partial x}=0\\
\dfrac{\partial p_{H_l\boldsymbol{\Theta}}\left(h_l,\boldsymbol{\theta},t\right)}{\partial t}+\dot{H}_l\left(\boldsymbol{\theta},t\right)\dfrac{\partial p_{H_l\boldsymbol{\Theta}}\left(h_l,\boldsymbol{\theta},t\right)}{\partial h_l}=-\mathscr{H}\left[H_l(\boldsymbol{\theta},t)\right]p_{H_l\boldsymbol{\Theta}}\left(h_l,\boldsymbol{\theta},t\right)\\
p_{H_l\boldsymbol{\Theta}}\left(h_l,\boldsymbol{\theta},t\right)\Big|_{t=t_0}=\delta\left(h_l-h_{l0}\right)p_{\boldsymbol{\Theta}}\left(\boldsymbol{\theta}\right)\\
\qquad l=1,2,\cdots,n
\end{cases}
\tag{43}
$$

where

$$
\mathscr{H}\left[H_l(\boldsymbol{\theta},t)\right]=\begin{cases}0 & H_l(\boldsymbol{\theta},t)\le[h_l]\\ 1 & H_l(\boldsymbol{\theta},t)>[h_l]\end{cases}
\tag{44}
$$

where $[h]$ is demand water pressure at the node No. l.

Different from the structural global reliability analysis, the functional reliability analysis of water supply network requires solving the probability density evolution equation for each node of the network. Miao et al presented a case study for a small-size network shown in Figure 8 (Miao, Liu & Li, 2018). In this network, all pipes are grey cast iron pipes. The length of pipe segments is 6 m. The network is located in type-II site and the soil is soft clay with the undrained shear strength of 22.93 kPa. The stiffness of axial and lateral soil springs can be gained according to the ALA seismic guidelines. Then based on the above analytical equations, the PDFs of the dynamic water head can be derived. Figure 9 shows the probability density surface and the probability density contour of the water pressure at a specific node. Figure 10 shows the cumulative probability density (CDF) of the water pressure at the node, where the comparative curves between physical equations invoked by non-steady flow (dynamic) and by steady flow (steady) are included, which prove the value of physical equations in the probability density evolution of stochastic systems.

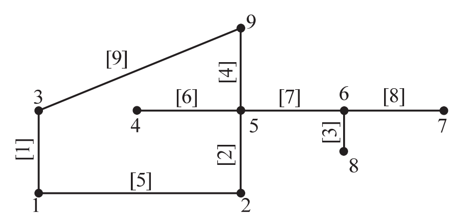
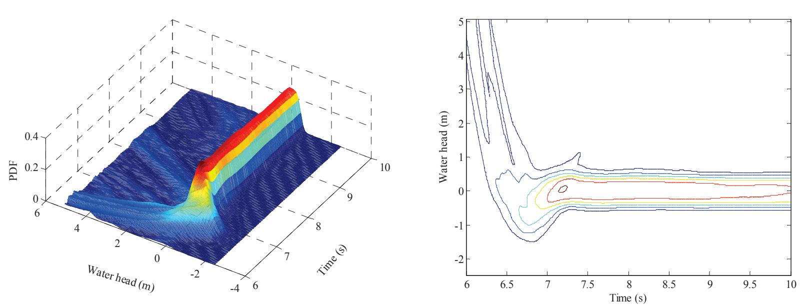


Figure 8. Schematic of a small-size pipe network.



(a) probability density surface (b) probability density contour

Figure 9. Probability density evolution of water head at the node No. 7.

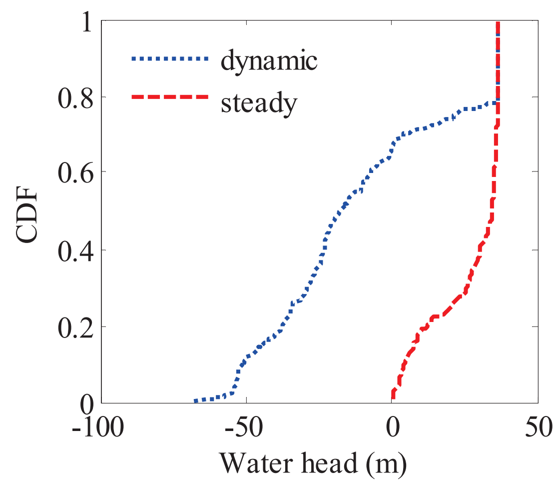


Figure 10. Cumulative probability density (CDF) of the water pressure at the node No. 7.

5 Conclusions and remarks

The research on the reliability of engineering structures has experienced a developing history for more than 90 years. A series of excellent scientific supposes, innovations and explorations were born in this process. Physically based system reliability research, as a new approach, will serve as a new link in this historical process. Connecting different types of physical equations with general probability density evolution equation, to give a set of basic governing equation for stochastic systems, provides a broad possibility for exploring the analysis, design and control of stochastic systems in different research fields. The cases provided in this paper could be viewed as several starting points associated with the new developing path invoked by the principle. We believe without any doubts that following this path, not only the third-generation design theory of engineering structures can be established, but also the initiative and academic self-consciousness can be gained in the process of scientifically recognizing and reflecting the objective world.

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