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A New Collocation Points Selection Strategy for Stochastic Response Surface Method

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The stochastic response surface method (SRSM) is widely used in engineering reliability analyses for its efficiency and accuracy. Selection of the collocation points in SRSM is always of great significance, as it may strongly affect the computed reliability results. An improved collocation method that achieves a better tradeoff between efficiency and accuracy is proposed, based on the merits of existing methods. A procedure for step-by-step implementation of the new method is provided. A simple case is employed to test the performance of the proposed method; and a comparative study is conducted to demonstrate advantages of the proposed method with respect to existing collocation methods.

Keywords: Reliability analysis, Stochastic response surface method, Collocation points, Symmetric full rank collocation method

1 Introduction

Recently, considerable efforts have been put on stochastic response surface method (SRSM) to solve reliability problems in structural and geotechnical engineering. For instance, Huang et al. (2007) developed a set of Excel add-in procedures to boost the application of SRSM without the need for advanced mathematical and programming skills; Li et al. (2011) proposed an improved SRSM for reliability analysis of rock slope involving correlated non-normal random variables; Mollon et al. (2011) applied the collocation-based SRSM to estimate the probability of face collapse in tunnels driven by a compressed-air pressurized shield; Wang and Li (2017) employed the SRSM to estimate the reliability of a tunnel excavation considering non-Gaussian dependent random variables under incomplete probability information.

Despite these efforts, however, one of the main challenges for the successful implementation of SRSM in practice is how to reliably and efficiently select collocation points to estimate the unknown coefficients of the polynomial chaos expansion (PCE), as it has a significant impact on the performance of SRSM (Li et al. 2011; Xiong et al. 2011). Previously, through a traditional regression-based collocation method (RBCM), Isukapalli et al. (1998) recommended that selected collocation points should be close to the origin, and be twice the number of unknown coefficients. However, the rank of the information matrix produced by RBCM cannot always fulfill a full rank, resulting in unstable and ineffective estimations (Li et al. 2011; Jiang et al. 2012; Li et al. 2013).

Alternatively, the linearly independent collocation method (LICM)-based SRSM can ensure a full rank information matrix, because it keeps rejecting collocation points until they fulfill such requirement; and the number of selected collocation points is equivalent to the number of unknown coefficients, thus being efficient (Jiang et al. 2012; Mao et al. 2012). However, the LICM may select highly asymmetrical collocation points with respect to the origin, leading to an unexpected result: higher odd-order SRSMs produce worse estimates of probability of failure than lower even-order SRSMs, as the probability density function is symmetric with respect to the origin (Xiao et al. 2014). To overcome this drawback, Xiao et al. (2014) proposed a new collocation method, but it requires many more collocation points than LICM. Therefore, a good strategy to select collocation points for SRSM, with a good balance between computational efficiency and accuracy, would be a welcome contribution to the field.

Building on the merits of LICM and RBCM, this paper proposes one new collocation points selection strategy to provide a better estimate of probability of failure using SRSM. Details of the algorithm are illustrated first, and then the proposed method is compared to other two commonly employed collocation methods using an illustrative case.

2 Stochastic response surface method

In the context of the stochastic response surface method (SRSM), the output can be represented in terms of Hermite polynomial chaos expansion (PCE) through a series of standard normal random variables (Isukapalli et al. 1998; Li et al. 2011):

$$\begin{aligned} y = {} & a_0\Gamma_0 + \sum_{i_1=1}^{n} a_{i_1}\Gamma_1(U_{i_1}) + \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2(U_{i_1},U_{i_2}) + \\ & \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}\Gamma_3(U_{i_1},U_{i_2},U_{i_3}) + \cdots + \\ & \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\cdots\sum_{i_n=1}^{i_{n-1}} a_{i_1 i_2,\cdots,i_n}\Gamma_n(U_{i_1},U_{i_2},\cdots,U_{i_n}) \end{aligned} \tag{1}$$

where y is the true response output provided by the deterministic model; $\Gamma_n(U_{i_1},U_{i_2},\ldots,U_{i_n})$ are multidimensional Hermite polynomials of order n; and $a_{i_1 i_2,\ldots,i_n}$ are unknown coefficients in the expansion, which can be estimated using the model outputs at selected collocation points (details of collocation points selection will be illustrated in Section 3). The number of the unknown coefficients, N_a, for a p^{th} order Hermite PCE involving n random variables is calculated by (Mollon et al. 2011)

$$N_a = \frac{(n+p)!}{n!\,p!} \tag{2}$$

3 A proposed symmetric full rank collocation method

In our proposed symmetric full rank collocation method (SFRCM), two types of collocation methods —RBCM and LICM— are combined to achieve a better balance between computational efficiency and accuracy. In particular, the new method combines the merits of RBCM (i.e., to obtain collocation points closer to the origin, and to maintain the symmetry of selected collocation points) with, the full rank criterion used in LICM (i.e., to ensure that the information matrix, **T**, is always invertible). A detailed implementation procedure is illustrated below to further explain our proposed method (see also the flowchart in Figure 1). The steps are:

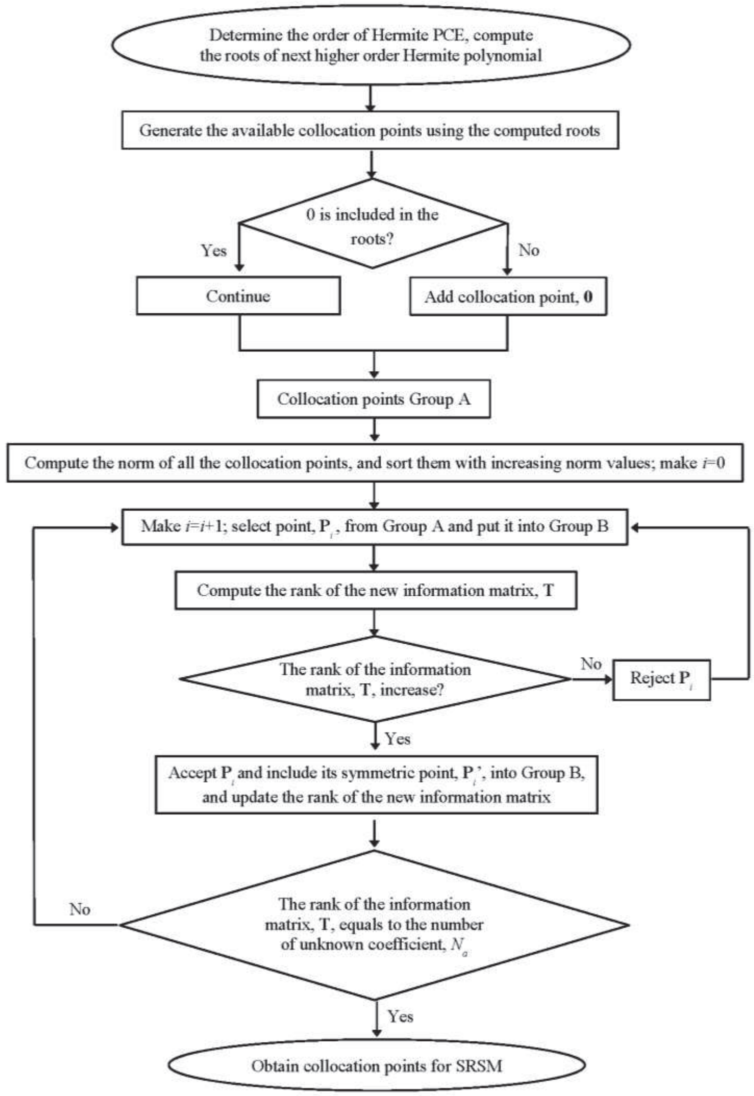

Figure 1. A flowchart to illustrate the implementation procedure of the proposed SFRCM

(i) Determine the order of Hermite PCE, and compute the roots of the next higher order Hermite polynomial. For instance, for a 2nd order Hermite PCE, the three roots of the 3rd order Hermite polynomial are 0, $\sqrt{3}$ and $-\sqrt{3}$.

(ii) Generate the available collocation points using the roots of the next higher order Hermite polynomial. If the Hermite polynomials that do not include a root of 0 are encountered, an additional collocation point located on the origin, **0**, should be included. Thus, the available collocation points Group A will contain $(p+1)^n$ or $(p+1)^n+1$ points.

(iii) Compute norms of all the collocation points, and sort them with increasing norm values. Make $i = 0$.
(iv) Make $i = i + 1$, select the collocation point, $\mathbf{P}_i$, from the available collocation points Group A, and put it into the selected collocation points Group B.
(v) Compute the rank of the new information matrix, **T**. If the rank of the information matrix does not increase, reject the collocation point, $\mathbf{P}_i$; otherwise, accept it.
(vi) If the collocation point, $\mathbf{P}_i$, is reserved in the selected collocation points Group B, its symmetric point with respect to the origin, $\mathbf{P}_i'$, should be included into Group B as well. Then, update the rank of the new information matrix, and delete $\mathbf{P}_i'$ from Group A.
(vii) Repeat Steps 4-6 until the rank of the information matrix, **T**, equals to the number of unknown coefficient, N_a.

In this way, our proposed SFRCM can guarantee that all the selected collocation points are symmetric with respect to the origin, and that the information matrix is always invertible. By following the selecting procedures, the number of selected collocation points will range between (N_a, $2N_a$), i.e., it is larger than that of LICM, but smaller than that of RBCM. The associated computational cost and accuracy will be illustrated in the case study.

4 Case study

This example, adapted from Michael and John (2009) with minor modifications, considers a simple uniform cantilever beam. Six parameters representing the material properties, loads and sizes of the beam are considered as independent random variables; their statistical information are given in Table 1. The limit state function due to exceedance of the displacement at the free end of the beam can be written as:

$$g(\mathbf{x}) = D_0 - \frac{4L^3}{Ewt}\sqrt{\left(\frac{Y}{t^2}\right)^2 + \left(\frac{X}{w^2}\right)^2} \tag{3}$$

where L is the length of the beam with a value of 2.54m, and D_0 = 6cm is the allowable displacement at the tip of the beam.

Table 1. Statistical information of random variables for the illustrative example

Random variable	Yield strength, R	Young's modulus, E	Horizontal load, X	Vertical load, Y	Width, w	Thickness, t
Units	MPa	GPa	N	N	cm	cm
μ^a	276	200	2224	4448	10	10
SD[b]	14	10	445	890	2	2

[a] μ = Mean value
[b] SD = standard deviation

The required number of collocation points, and the corresponding computed probabilities of failure, are listed in Table 2. It should be noted that other existing collocation methods (i.e., RBCM and LICM) are employed herein for comparison, and that Monte Carlo simulation (MCS) is also conducted using the true limit state functions (LSFs) to obtain unbiased estimates of the reliability result, as the 'reference' or 'exact' solution for comparison.

The number of collocation points, N_p, required by our proposed SFRCM is between those employed by LICM (N_a) and RBCM ($2N_a$). If we define a ratio representing N_p divided by $1.5N_a$, it will be approximately 1.23, 1.02, 0.99 and 0.94, respectively, for PCEs of orders 1-4.

This indicates that the required number of collocation points by SFRCM will vary from above $1.5N_a$ to below $1.5N_a$ with increasing PCEs orders for the case considered.

Regarding the computed reliability results, the proposed SFRCM can often outperform LICM. In particular, when the 3rd order PCE is applied, the probability of failure, P_f, computed by LICM, has a relative error of -17.26% with respect to the MCS result; however, for SFRCM, the relative error is only 1.45%, which is typically considered acceptable in engineering practice. This, again, reveals the influence of the asymmetrical collocation points employed by LICM for the 3rd order on the computed reliability result. As for RBCM, all orders of PCEs used herein (orders 1-4) fail to obtain reasonable reliability results, since all information matrices cannot fulfill the requirement of full rank.

Table 2. Computed results by different collocation methods for the illustrative example

$n = 6$	Order	N_p [a]	$P_f(10\times^{-2})$	Δ (%) [b]
LICM	1	7	0.01	-99.75
	2	28	1.64	-38.36
	3	84	2.20	-17.26
	4	210	2.65	-0.20
SFRCM	1	13	0.05	-98.21
	2	43	2.11	-20.52
	3	125	2.69	1.45
	4	295	2.69	1.42
RBCM	1	14	0.00	-100.00
	2	56	0.00	-100.00
	3	168	0.00	-100.00
	4	420	49.97	1782.54
MCS	-	5,000,000	2.65 [c]	-

[a] N_p=Number of limit state function evaluations.
[b] Δ=Relative error in relation to MCS.
[c] COV_{P_f}= 0.27%

5 Summary and conclusions

This paper proposes an improved symmetric full rank collocation method (SFRCM) to select collocation points employed in the stochastic response surface method (SRSM) for structural reliability analysis. To achieve a good balance between computational efficiency and accuracy, the proposed method makes an integration of the main features of two conventional collocation methods: LICM and RBCM. And then an illustrative example is used to demonstrate the advantages of the proposed method with respect to these two classical alternative approaches. The main conclusions of such analysis can be summarized as follows.

(1) RBCM using twice number of unknown coefficients cannot always ensure robust reliability result since the information matrix has not always a full rank;

(2) LICM demanding collocation points with an identical quantity of unknown coefficients inherently ensures a full rank information matrix. However, when the 3rd order PCE is applied, LICM leads to a significant error (compared to the proposed SFRCM), which is mainly attributed to the asymmetry of the selected collocation points.

(3) Our proposed method SFRCM achieves the best overall performance. With merely a few more collocation points than LICM, SFRCM can simultaneously achieve a full rank information

matrix and the symmetry of selected points, thereby resulting in a more robust estimation of the probability of failure.

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