International Journal of Aerospace and Lightweight Structures
Vol. 1, No. 1 (2011) 89–107

DOI: 10.3850/S2010428611000031

NONLINEAR THERMAL BENDING FOR SHEAR DEFORMABLE NANOBEAMS BASED ON NONLOCAL ELASTICITY THEORY

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Accepted 1 September 2011

Nonlinear bending of shear deformable nanobeams subject to a temperature field is investigated in this paper based on von Kármán type nonlinearity and nonlocal elasticity theory. By using the variational principle approach, new higher-order governing differential equations and the corresponding higher-order boundary conditions both in the transverse and axial directions are derived and discussed. Several examples are presented to highlight the effects of nonlocal nanoscale, temperature and shear deformation on the transverse deflection of nanobeam. The exact analytical solutions for transverse deflection are derived and the solutions confirm that the nonlocal nanoscale tends to significantly decrease nanobeam transverse deflection while shear deformation increases the transverse deflection of nanobeam. It is also concluded that the stiffness of shear deformable nanobeams could be reinforced at low and room temperature, while at high temperature the stiffness will be reduced.

Keywords: Nonlocal elasticity, Thermal-elasticity, Shear deformable nanobeam, Variation principle, Thermal effects.

1. Introduction

With the advent of nanoengineering and nanotechnology, nanostructures including nanotube, nanobeam and nanorod etc., have become potential design candidates which are likely to play key roles in many engineering devices or components at the nanometer scale, such as micro- or nano-electromechanical systems (MEMS or NEMS). Recently, a great deal of research indicates that the material properties of nanostructures are related to temperature change, subsequently, there were numerous researches which concerned thermal bending, vibration and buckling analyses of nanostructures. At nanoscale, the mechanical properties of nanostructures are significantly different from their behavior at macroscopic due to the inherent size effects. There have been some essential approaches to investigate the mechanical properties of nanostructures considering size and thermal effects, including, but not restricted to, strain gradient models [Mindlin, 1964, 1965], couple stress models [Mindlin, 1962; Toupin, 1962], and nonlocal stress models [Eringen, 1972, 1972, 1972, 1981, 1983]. The methods would potentially play a critical part in the analyses related to nanotechnological applications.

The nonlocal elasticity field theory used in this paper, which was first developed by Eringen [1972, 1972, 1972, 1981, 1983] and his associates, assumes that the stress tensor at a point is a function of strains at all points in the continuum. It is different from the classical continuum theory because the latter is based on a constitutive relation which states that the stress at a point is a function of strain at that particular point. This nonlocal theory is proved to be in accordance with atomic model of lattice dynamics and with experimental observations on phonon dispersion. At present, it has been extensively applied to analyze bending, buckling, vibration and wave propagation of CNTs and other nanostructures.

Based on this model, Murmu and Pradhan [Murmu *et al.*, 2009, 2010] investigated the buckling and vibration of single-walled CNTs with thermal effect. Wang and his associates [Wang *et al.*, 2006, 2009] investigated buckling and postbuckling of micro and nanorods by applying the nonlocal beam theory considering shear deformation and nonlinearity respectively. Later, based on the nonlocal Timoshenko beam model, Benzair *et al.*, [2008] studied free vibration of single-walled nanotubes with thermal effect. Li and Kardonatea [2007] investigated the thermal buckling phenomenon of multi-walled CNTs in an elastic medium using the nonlocal theory. Tounsi *et al.*, [2008] investigated the small size effect on wave propagation in double-walled CNTs under temperature field. Yan *et al.*, [2010] studied the small scale effect on the buckling behaviors of triple-walled CNTs with the initial axial stress under temperature field. However, the nonlocal models used in these references [Murmu *et al.*, 2009, 2010; Wang *et al.*, 2006, 2009; Benzair *et al.*, 2008; Li *et al.*, 2007; Tounsi *et al.*, 2008; Yan *et al.*, 2010] are termed the partial nonlocal models and they, in fact, do not satisfy the conditions of equilibrium [Lim, 2009, 2010, 2010] and do not describe the true state of motion. These references reached a common conclusion that, in many cases, the buckling load or frequency of nanostructures reduces in the presence of nonlocal nanoscale effects. This conclusion is surprising and contradictory to intrinsic intuition, as nanoscale size effects are believed to enhance nanostructural stiffness and lead to higher buckling load and natural frequencies. This contradictory technical query is exactly the key issue of this paper.

In the application of nonlocal elasticity models for nanostructure, Lim [2009, 2010, 2010] recently showed that the classical governing equations of motion and equilibrium equations cannot be directly applied in nonlocal stress models even with the relevant quantities replaced by the corresponding nonlocal quantities. He proposed a new nonlocal stress model which considers the nonlinear history of finite straining in the derivation of the strain energy density and further derived exact equilibrium conditions and higher-order differential governing equations with the corresponding higher-order nonlocal boundary conditions via the variational principle in which an effective nonlocal bending moment is derived as an infinite series of higher-order nonlocal bending moments. Using the new exact nonlocal model, Lim and his associates further analyzed buckling [Lim *et al.*, 2010], vibration [Li *et al.*, 2011], wave propagation [Lim *et al.*, 2010, 2010; Yang *et al.*, 2011], thermal bending [Lim *et al.*, 2011] and thermal buckling [Yang *et al.*, 2011] of nanostructures.

Applying this exact stress model, this paper investigated the nonlinear thermal bending of nanobeam based on Timoshenko beam model and with von Kármán type geometric nonlinearity. New higher-order differential equations of equilibrium with the corresponding non-classical boundary conditions for nonlinear thermal bending of nanobeam, which include essential higher-order nonlocal terms missing in the previous partial nonlocal stress models, are derived. Analytical solutions for some practical examples with various boundary conditions are obtained and discussed in detail. Conclusions are consequently drawn that stiffness of shear deformable nanobeam increases with increasing nonlocal effect while it decreases with the effect of shear deformation. The paper also concludes that at low and room temperature the nanobeam thermal-elastic deflection decreases with increasing temperature difference, while at high temperature the deflection increases as the temperature difference increases.

2. Nonlinear Thermal Elastic Model for Shear Deformable Nanobeam

According to the Timoshenko beam theory and considering von Kármán type nonlinearity for large deflection, the strain-displacement relations are given by

$$\varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - z\frac{d\varphi}{dx} \tag{1}$$

$$\gamma_{xz} = \frac{dw}{dx} - \varphi = w^{\langle 1 \rangle} - \varphi \tag{2}$$

where u and w are the axial and transverse deflections of nanobeam neutral axis, φ is rotation of nanobeam cross section, x is the axial coordinate measured from the left end and z is the normal coordinate measured from the midplane, as shown in Fig. 1.

In accordance with the nonlocal elastic stress theory of Eringen [1983], the nonlocal constitutive equation for nanobeam considering nonlocal effect can be expressed as

$$\sigma_{xx} - (e_0 a)^2 \frac{d^2\sigma_{xx}}{dx^2} = E\varepsilon_{xx} \tag{3}$$

where a is an internal characteristic length (e.g., lattice parameter, C-C bond length, granular distance, etc.), e_0 is a material constant, σ_{xx} is the nonlocal normal stress, E is the classical Young's modulus, and ε_{xx} is the normal strain. The magnitude of e_0 is determined experimentally or approximated by matching the dispersion

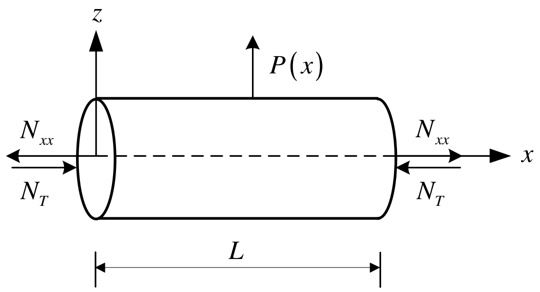

Fig. 1. Coordinate system of a shear deformable nanobeam.

curves of plane waves with those of atomic lattice dynamics. The normal strain can be expressed as

$$\varepsilon_{xx} = \varepsilon_m + \varepsilon_b \tag{4}$$

where ε_m is the axial strain and ε_b is the bending strain.

For generality, the equation above is non-dimensionalized using the following dimensionless parameters

$$\bar{\sigma}_{xx} = \frac{\sigma_{xx}}{E}\ ,\ \bar{z} = \frac{z}{L}\ ,\ \bar{x} = \frac{x}{L}\ ,\ \bar{w} = \frac{w}{L}\ ,\ \bar{u} = \frac{u}{L}\ ,\ \tau = \frac{e_0 a}{L} \tag{5}$$

Hence, Eq. (3) can be expressed as

$$\bar{\sigma}_{xx} - \tau^2 \frac{d^2\bar{\sigma}_{xx}}{d\bar{x}^2} = \varepsilon_m + \varepsilon_b \tag{6}$$

where

$$\varepsilon_m = \bar{u}^{\langle 1\rangle} + \frac{1}{2}\left(\bar{w}^{\langle 1\rangle}\right)^2 \tag{7}$$

$$\varepsilon_b = -\bar{z}\,\frac{d\varphi}{d\bar{x}} = -\bar{z}\varphi^{\langle 1\rangle} \tag{8}$$

in which $()^{\langle n\rangle} = d^n/d\bar{x}^n$ represents the derivative with respect to the dimensionless coordinate $\bar{x}$. Neglecting all pre-stresses and pre-strains which appear as constants of integration, the solution to Eq. (6) can be expressed as

$$\bar{\sigma}_{xx} = \sum_{n=1}^{\infty} \tau^{2(n-1)} \varepsilon_{xx}^{\langle 2(n-1)\rangle} \tag{9}$$

which relates the nonlocal stress and strain gradients. Considering only the most significant nonlocal term with $n = 2$, Eq. (9) can be reduced to

$$\bar{\sigma}_{xx} = (\varepsilon_m + \varepsilon_b) + \tau^2\left(\varepsilon_m + \varepsilon_b\right)^{\langle 2\rangle} \tag{10}$$

The strain energy density e at a point in the nanostructure is the integral sum of the nonlocal stress over the history of straining which can be expressed as

$$e = \int_0^{\varepsilon_{xx}} \sigma_{xx}\, d\varepsilon_{xx} = E\int_0^{\varepsilon_{xx}} \bar{\sigma}_{xx}\, d\varepsilon_{xx} \tag{11}$$

Expanding Eq. (11) using the constitutive solution in Eq. (10) yields

$$e = e_1 + e_2 \tag{12}$$

where

$$\begin{aligned} e_1 &= \tfrac{1}{2}E\left(\varepsilon_m + \varepsilon_b\right)^2 \\ e_2 &= \tfrac{1}{2}E\tau^2\left[\left(\varepsilon_m + \varepsilon_b\right)^{\langle 1\rangle}\right]^2 \end{aligned} \tag{13}$$

Furthermore, the energy density contributed by shear stress should also be considered for a shear deformable nanobeam as

$$e_s = \frac{1}{2}G\gamma_{xz}^2 \tag{14}$$

where G is the shear modulus and γ_{xz} is shear strain and the shear force is defined as

$$Q = \int\limits_A G\gamma_{xz}dA = \int\limits_A G\left(\bar{w}^{\langle 1\rangle} - \varphi\right)dA = GA\kappa_s\left(\bar{w}^{\langle 1\rangle} - \varphi\right) \tag{15}$$

in which κ_s is the shear correction factor. Thus the total strain energy in the whole deformed body with volume V is

$$U = \int_V \left(e + e_s\right)dV = \int_V \left(e_1 + e_2 + e_s\right)dV \tag{16}$$

3. Higher-Order Nonlocal Governing Equations and Boundary Conditions

The variational principle is applied to determine the governing equation of equilibrium and boundary conditions. Variation of the strain energy in Eq. (16) yields

$$\begin{aligned} \delta U = \textstyle\int_0^1 &\left[EAL\left(-\varepsilon_m^{\langle 1\rangle} + \tau^2\varepsilon_m^{\langle 3\rangle}\right)\bar{w}^{\langle 1\rangle} + EAL\left(-\varepsilon_m + \tau^2\varepsilon_m^{\langle 2\rangle}\right)\bar{w}^{\langle 2\rangle}\right. \\ &\left.-GAL\kappa_s\left(\bar{w}^{\langle 2\rangle} - \varphi^{\langle 1\rangle}\right)\right]\delta\bar{w}d\bar{x} \\ &+\textstyle\int_0^1 EAL\left(-\varepsilon_m^{\langle 1\rangle} + \tau^2\varepsilon_m^{\langle 3\rangle}\right)\delta\bar{u}d\bar{x} \\ &+\textstyle\int_0^1 \left[\frac{EI}{L}\left(\tau^2\varphi^{\langle 4\rangle} - \varphi^{\langle 2\rangle}\right) - GAL\kappa_s\left(\bar{w}^{\langle 1\rangle} - \varphi\right)\right]\delta\varphi d\bar{x} \\ &+\left\{\left[EAL\left(\varepsilon_m - \tau^2\varepsilon_m^{\langle 2\rangle}\right)\bar{w}^{\langle 1\rangle} + GAL\kappa_s\left(\bar{w}^{\langle 1\rangle} - \varphi\right)\right]\delta\bar{w}\right. \\ &+EAL\tau^2\varepsilon_m^{\langle 1\rangle}\bar{w}^{\langle 1\rangle}\delta\bar{w}^{\langle 1\rangle} \\ &+EAL\left(\varepsilon_m - \tau^2\varepsilon_m^{\langle 2\rangle}\right)\delta\bar{u} + EAL\tau^2\varepsilon_m^{\langle 1\rangle}\delta\bar{u}^{\langle 1\rangle} \\ &\left.+\tfrac{EI}{L}\left(\varphi^{\langle 1\rangle} - \tau^2\varphi^{\langle 3\rangle}\right)\delta\varphi + \tfrac{EI}{L}\tau^2\varphi^{\langle 2\rangle}\delta\varphi^{\langle 1\rangle}\right\}_0^1 \end{aligned} \tag{17}$$

For static bending of a nanobeam subject to transverse load and axial tension, the work W_1 exerted by external distributed transverse load $P(x)$ and axial tension

N_{xx} at the two ends is

$$W_1 = LN_{xx}\, \bar{u}|_{x=0}^{x=1} - \frac{LN_{xx}}{2} \int_0^1 \left(\bar{w}^{\langle 1 \rangle}\right)^2 d\bar{x} + L^2 \int_0^1 P\bar{w}\, d\bar{x} \tag{18}$$

It is assumed that the thermal diffusion processes in the nonlocal continuum are of local nature [Ardito *et al.*, 2009; Polizzotto, 2001, 2003]. Therefore, in this paper, the effect of temperature rise is considered as an additional force in the axial direction. As a result of thermal expansion, the additional force is given by

$$N_T = \alpha TEA \tag{19}$$

where T is the temperature difference with respect to an initial reference temperature, A is the cross sectional area and α is the thermal expansion coefficient. The work done by this axial force N_T is

$$W_2 = \frac{N_T}{2} \int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx = \frac{\alpha TEAL}{2} \int_0^1 \left(\frac{\partial \bar{w}}{\partial \bar{x}}\right)^2 d\bar{x} \tag{20}$$

Hence, the variation of work done on the nanobeam is

$$\delta W_1 = LN_{xx}\, \delta\bar{u}|_0^1 - LN_{xx}\bar{w}^{\langle 1 \rangle}\, \delta\bar{w}|_0^1 + LN_{xx} \int_0^1 \bar{w}^{\langle 2 \rangle} \delta\bar{w}\, d\bar{x} + L^2 \int_0^1 P\, \delta\bar{w}\, d\bar{x} \tag{21}$$

$$\delta W_2 = LN_T\bar{w}^{\langle 1 \rangle}\, \delta\bar{w}|_0^1 - LN_T \int_0^1 \bar{w}^{\langle 2 \rangle} \delta\bar{w}\, d\bar{x} \tag{22}$$

According to the variational principle, variation of the functional $U - W_1 - W_2$ vanishes, or

$$\delta\left(U - W_1 - W_2\right) = 0 \tag{23}$$

in the search for an extremum. Since $\delta\bar{w}$, $\delta\bar{u}$ and $\delta\varphi$ do not vanish according to the variational principle, Eq. (23) yields the higher-order nonlocal equilibrium equations as

$$\begin{aligned} &EAL\left(-\varepsilon_m^{\langle 1 \rangle} + \tau^2\varepsilon_m^{\langle 3 \rangle}\right)\bar{w}^{\langle 1 \rangle} + EAL\left(-\varepsilon_m + \tau^2\varepsilon_m^{\langle 2 \rangle}\right)\bar{w}^{\langle 2 \rangle} - GAL\kappa_s\left(\bar{w}^{\langle 2 \rangle} - \varphi^{\langle 1 \rangle}\right) \\ &+ \left(LN_T - LN_{xx}\right)\bar{w}^{\langle 2 \rangle} = L^2P \end{aligned} \tag{24}$$

$$\frac{EI}{L}\left(\tau^2\varphi^{\langle 4 \rangle} - \varphi^{\langle 2 \rangle}\right) - GAL\kappa_s\left(\bar{w}^{\langle 1 \rangle} - \varphi\right) = 0 \tag{25}$$

$$EAL\left(-\varepsilon_m^{\langle 1 \rangle} + \tau^2\varepsilon_m^{\langle 3 \rangle}\right) = 0 \tag{26}$$

From Eq. (26), $-\varepsilon_m + \tau^2\varepsilon_m^{\langle 2 \rangle}$ is a constant and, upon integration with respect to $\bar{x}$, it can be shown that

$$N_\varepsilon = EA\left(-\varepsilon_m + \tau^2\varepsilon_m^{\langle 2 \rangle}\right) \tag{27}$$

where N_ε is the total compression load induced by the presence of axial strain in the nanobeam. In the axial direction, the nanobeam is fixed at $\bar{x} = 0$ and free at

$\bar{x}=1$. Hence, from the boundary conditions, we see that

$$N_{\varepsilon}=-N_{xx}+N_T \tag{28}$$

Substitute Eqs. (26) and (27) into Eq. (24), the differential governing equation can be expressed as

$$LP_{xx}\bar{w}^{\langle 2\rangle}-GAL\kappa_s\left(\bar{w}^{\langle 2\rangle}-\varphi^{\langle 1\rangle}\right)=L^2P \tag{29}$$

where P_{xx} is the total force in axial direction and can be expressed as $P_{xx}=2\left(N_T-N_{xx}\right)$. In this paper, the bending of nanobeam under a combined action of axial tension load and transverse load under thermal field is most interested. Therefore, $-P_{xx}=2\left(N_{xx}-N_T\right)>0$ is considered.

Similarly, the corresponding higher-order nonlocal boundary conditions obtained from the remaining terms in Eq. (23) can be shown as follows

$$\begin{aligned}&\left[EAL\left(\varepsilon_m-\tau^2\varepsilon_m^{\langle 2\rangle}\right)\bar{w}^{\langle 1\rangle}+GAL\kappa_s\left(\bar{w}^{\langle 1\rangle}-\varphi\right)+\left(LN_{xx}-LN_T\right)\bar{w}^{\langle 1\rangle}\right]_{\bar{x}=0,1}=0\\ &\text{or}\quad \bar{w}|_{\bar{x}=0,1}=0\end{aligned} \tag{30}$$

$$\left.\begin{aligned}&\tfrac{EI}{L}\left(\varphi^{\langle 1\rangle}-\tau^2\varphi^{\langle 3\rangle}\right)=0 \quad &\text{or}\quad &\varphi=0\\ &\tfrac{EI}{L}\tau^2\varphi^{\langle 2\rangle}=0 &\text{or}\quad &\varphi^{\langle 1\rangle}=0\end{aligned}\right\}\text{at } \bar{x}=0,1 \tag{31}$$

$$\left.\begin{aligned}&EAL\left(\varepsilon_m-\tau^2\varepsilon_m^{\langle 2\rangle}\right)=LN_{xx}-LN_T \quad &\text{or}\quad &\bar{u}=0\\ &EAL\tau^2\varepsilon_m^{\langle 1\rangle}=0 &\text{or}\quad &\bar{u}^{\langle 1\rangle}=0\end{aligned}\right\}\text{at } \bar{x}=0,1 \tag{32}$$

The displacement and rotation in Eqs. (24), (25) and (26) for thermal buckling of shear deformable nanobeam can be decoupled as

$$\tau^2\varphi^{\langle 5\rangle}-\varphi^{\langle 3\rangle}+\beta\varphi^{\langle 1\rangle}=-\bar{P}\gamma \tag{33}$$

$$\tau^2\bar{w}^{\langle 6\rangle}-\bar{w}^{\langle 4\rangle}+\beta\bar{w}^{\langle 2\rangle}=-\bar{P}\gamma \tag{34}$$

where $\beta=\frac{L^2P_{xx}}{EI}\frac{GAL\kappa_s}{GAL\kappa_s+LP_{xx}}$, $\gamma=\frac{GAL\kappa_s}{GAL\kappa_s+LP_{xx}}$ and $\bar{P}=\frac{PL^3}{EI}$. Considering $\beta>0$ and $\tau<1$, then $0<\Delta=1-4\tau^2\beta<1$ can be deduced. Therefore the general solution to Eq. (34) comprises both the homogeneous solution and the particular solution which can be expressed as

$$\bar{w}=C_1e^{\lambda_1\bar{x}}+C_2e^{-\lambda_1\bar{x}}+C_3e^{\lambda_2\bar{x}}+C_4e^{-\lambda_2\bar{x}}+C_5\bar{x}+C_6-\frac{\bar{P}\gamma}{2\beta}\bar{x}^2 \tag{35}$$

with six constants of integration which can be determined from the boundary conditions for $\bar{w}$ in Eq. (30) and for φ in Eq. (31) and

$$\lambda_1 = \sqrt{\frac{1+\sqrt{1-4\tau^2\beta}}{2\tau^2}} \quad ; \quad \lambda_2 = \sqrt{\frac{1-\sqrt{1-4\tau^2\beta}}{2\tau^2}} \tag{36}$$

Substituting Eq. (35) into Eq. (29) and integrating once, φ can be expressed as

$$\varphi = \frac{1}{\gamma}\left(C_1\lambda_1 e^{\lambda_1\bar{x}} - C_2\lambda_1 e^{-\lambda_1\bar{x}} + C_3\lambda_2 e^{\lambda_2\bar{x}} - C_4\lambda_2 e^{-\lambda_2\bar{x}}\right) + C_7 - \frac{\bar{P}\bar{x}}{\beta} + \frac{L^2 P\bar{x}}{GAL\kappa_s} \tag{37}$$

Further, substituting Eq. (35) and Eq. (37) into Eq. (25), one obtains

$$C_5 = C_7 \tag{38}$$

Hence, the transverse rotation φ can be finally expressed as

$$\varphi = \frac{1}{\gamma}\left(C_1\lambda_1 e^{\lambda_1\bar{x}} - C_2\lambda_1 e^{-\lambda_1\bar{x}} + C_3\lambda_2 e^{\lambda_2\bar{x}} - C_4\lambda_2 e^{-\lambda_2\bar{x}}\right) + C_5 - \frac{\bar{P}\bar{x}}{\beta} + \bar{P}\eta\bar{x} \tag{39}$$

where $\eta = \frac{EI}{GAL^2\kappa_s} = \frac{E}{16G\kappa_s}\left(\frac{d}{L}\right)^2$ is related to material constants and diameter to length ratio d/L.

4. Examples and Discussion

Several examples for nanobeams with different boundary conditions are presented in this section to highlight the effect of nonlocal effect, shear deformation and temperature difference. The different boundary conditions considered include simply supported (SS), propped cantilever (CS), and clamp-clamp (CC) nanobeams. The material constants and parameters are $\nu = 0.19$, $E = 1.1$TPa, $G = E/2(1+\nu)$, shear correction factor $\kappa_s = 0.8$, diameter to length ratio $d/L = 0.05$ and negative thermal coefficient $\alpha = -1.6\times10^{-6}$ at low and room temperature, and positive thermal coefficient $\alpha = 1.1\times10^{-6}$ at high temperature [Jiang, 2004]. The nanobeams are subjected to distributed transverse load $P(x)$ as defined in the beginning of Sec. 3 and following Eq. (34). In the following examples, the boundary conditions in the z-direction vary but in the x-direction, all cases do have the left-end fixed and the right-end free.

4.1. *Simply Supported (SS) Shear Deformable Nanobeam*

For a nanobeam simply supported in the z-direction, the six higher-order boundary conditions in the transverse direction are

$$\left.\varphi^{\langle 1\rangle}\right|_{\bar{x}=0,1} = 0 \quad ; \quad \left.\varphi^{\langle 3\rangle}\right|_{\bar{x}=0,1} = 0 \quad ; \quad \left.\bar{w}\right|_{\bar{x}=0,1} = 0 \tag{40}$$

Substituting Eqs. (35) and (39) into Eq. (40), the constants of integration are

$$\begin{aligned} &C_1 = \frac{\bar{P}(\eta\beta-1)\gamma\lambda_2^2}{\left(1+e^{\lambda_1}\right)\beta\lambda_1^2\left(\lambda_2^2-\lambda_1^2\right)}\ ; C_2 = \frac{\bar{P}e^{\lambda_1}(\eta\beta-1)\gamma\lambda_2^2}{\left(1+e^{\lambda_1}\right)\beta\lambda_1^2\left(\lambda_2^2-\lambda_1^2\right)}\ ; C_3 = \frac{\bar{P}(\eta\beta-1)\gamma\lambda_1^2}{\left(1+e^{\lambda_2}\right)\beta\lambda_2^2\left(\lambda_2^2-\lambda_1^2\right)} \\ &C_4 = \frac{\bar{P}e^{\lambda_2}(\eta\beta-1)\gamma\lambda_1^2}{\left(1+e^{\lambda_2}\right)\beta\lambda_2^2\left(\lambda_2^2-\lambda_1^2\right)}\ ; \qquad C_5 = \frac{\bar{P}\gamma}{2\beta} \qquad ;\ C_6 = \frac{\bar{P}\gamma(\eta\beta-1)\left(\lambda_1^2+\lambda_2^2\right)}{\beta\lambda_1^2\lambda_2^2} \end{aligned} \tag{41}$$

Hence, the transverse deflection and rotation using a nonlocal Timoshenko model for a simply supported nanobeam subjected to a uniformly distributed load $P(x)$ are

$$\begin{aligned}\bar{w}_T = & \frac{\bar{P}\gamma}{2\beta}\bar{x} - \frac{\bar{P}\gamma}{2\beta}\bar{x}^2 + \frac{\bar{P}\gamma(\eta\beta-1)}{\beta}\left[\frac{\lambda_2^2\left(e^{\lambda_1\bar{x}}+e^{\lambda_1(1-\bar{x})}\right)}{\left(1+e^{\lambda_1}\right)\lambda_1^2\left(\lambda_1^2-\lambda_2^2\right)}\right. \\ & \left. -\frac{\lambda_1^2\left(e^{\lambda_2\bar{x}}+e^{\lambda_2(1-\bar{x})}\right)}{\left(1+e^{\lambda_2}\right)\lambda_2^2\left(\lambda_1^2-\lambda_2^2\right)} + \frac{\left(\lambda_1^2+\lambda_2^2\right)}{\lambda_1^2\lambda_2^2}\right]\end{aligned} \tag{42}$$

$$\varphi_T = \frac{\bar{P}\gamma}{2\beta} - \frac{\bar{P}\bar{x}}{\beta} + \bar{P}\bar{x}\eta + \frac{\bar{P}(\eta\beta-1)}{\beta}\left[\frac{\lambda_2^2\left(e^{\lambda_1\bar{x}}-e^{\lambda_1(1-\bar{x})}\right)}{\left(1+e^{\lambda_1}\right)\lambda_1\left(\lambda_1^2-\lambda_2^2\right)} + \frac{\lambda_1^2\left(e^{\lambda_2\bar{x}}+e^{\lambda_2(1-\bar{x})}\right)}{\left(1+e^{\lambda_2}\right)\lambda_2\left(\lambda_1^2-\lambda_2^2\right)}\right] \tag{43}$$

For vanishing axial force parameter $\beta \to 0$, Eq. (42) can be readily degenerated to the linear bending of a shear deformable nanobeam and can be easily shown as

$$\begin{aligned}(\bar{w}_T)_L = & \frac{\bar{P}}{24}\left[\left(\bar{x}-2\bar{x}^3+\bar{x}^4\right) - 12\tau^2\left(\bar{x}-\bar{x}^2\right) + 24\tau^4\left(1-\frac{e^{\bar{x}/\tau}+e^{(1-\bar{x})/\tau}}{1+e^{1/\tau}}\right)\right] \\ & + \frac{\bar{P}\eta}{2}\left(\bar{x}-\bar{x}^2\right)\end{aligned} \tag{44}$$

For $GA\kappa_s \to \infty$, Eq. (42) can be reduced to the nonlinear bending of a Euler-Bernoulli nanobeam, as

$$\begin{aligned}\bar{w}_E = & \frac{\bar{P}}{2\beta}\bar{x} - \frac{\bar{P}}{2\beta}\bar{x}^2 + \frac{\bar{P}(\eta\beta-1)}{\beta}\left[\frac{\lambda_2^2\left(e^{\lambda_1\bar{x}}+e^{\lambda_1(1-\bar{x})}\right)}{\left(1+e^{\lambda_1}\right)\lambda_1^2\left(\lambda_1^2-\lambda_2^2\right)}\right. \\ & \left. -\frac{\lambda_1^2\left(e^{\lambda_2\bar{x}}+e^{\lambda_2(1-\bar{x})}\right)}{\left(1+e^{\lambda_2}\right)\lambda_2^2\left(\lambda_1^2-\lambda_2^2\right)} + \frac{\left(\lambda_1^2+\lambda_2^2\right)}{\lambda_1^2\lambda_2^2}\right]\end{aligned} \tag{45}$$

where $\beta = \frac{L^2 P_{xx}}{EI}$ in this equation. Eq. (42) can also be reduced to the transverse deflection for linear bending of a Euler-Bernoulli nanobeam

$$(\bar{w}_E)_L = \frac{\bar{P}}{24}\left[\left(\bar{x}-2\bar{x}^3+\bar{x}^4\right) - 12\tau^2\left(\bar{x}-\bar{x}^2\right) + 24\tau^4\left(1-\frac{e^{\bar{x}/\tau}+e^{(1-\bar{x})/\tau}}{1+e^{1/\tau}}\right)\right] \tag{46}$$

by setting $GA\kappa_s \to \infty$ and vanishing axial force parameter $\beta \to 0$, which is first solved by Lim [2010]. For vanishing nonlocal effect, $\tau \to 0$, Eq. (42) is reduced to the classical Timoshenko beam solution

$$\bar{w}_{T\mathrm{cla}} = \frac{\bar{P}\gamma}{2\beta}\bar{x} - \frac{\bar{P}\gamma}{2\beta}\bar{x}^2 + \frac{\bar{P}\gamma(\eta\beta-1)}{\beta^2}\left[1 - \frac{e^{\sqrt{\beta}(1-\bar{x})}}{\left(1+e^{\sqrt{\beta}}\right)\beta^2} - \frac{e^{\sqrt{\beta}\bar{x}}}{\left(1+e^{\sqrt{\beta}}\right)\beta^2}\right] \tag{47}$$

The maximum bending deflection $(\bar{w}_{Tcla})_{\max}$ occur at $\bar{x} = 1/2$ and it is

$$\left(\bar{w}_{T\mathrm{cla}}\right)_{\max} = \frac{\bar{P}\gamma}{8\beta} + \frac{\bar{P}\gamma(\eta\beta-1)}{\beta^2}\left[1 - \frac{2e^{\sqrt{\beta}/2}}{\left(1+e^{\sqrt{\beta}}\right)\beta^2}\right] \tag{48}$$

The nonlocal size effect τ on the dimensionless deflection ratio $\bar{w}_T/\left(\bar{w}_{T\mathrm{cla}}\right)_{\max}$ is demonstrated in Fig. 2, where $\bar{w}_T$ is the thermal-elastic deflection based on nonlocal Timoshenko beam model as expressed in Eq. (42) and $\left(\bar{w}_{T\mathrm{cla}}\right)_{\max}$ is the maximum

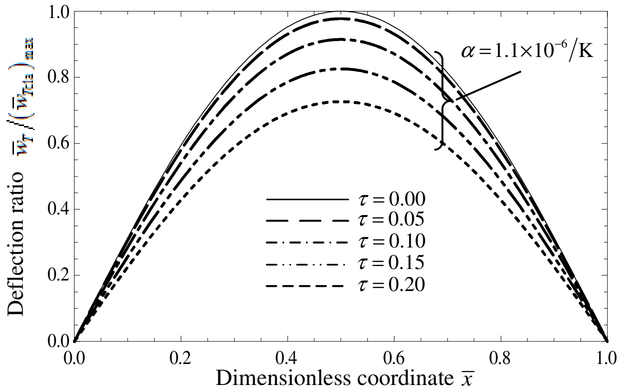

Fig. 2. The effect of nanoscale τ on $\bar{w}_T$ / $(\bar{w}_{T\mathrm{cla}})_{\max}$ for SS nanobeam.

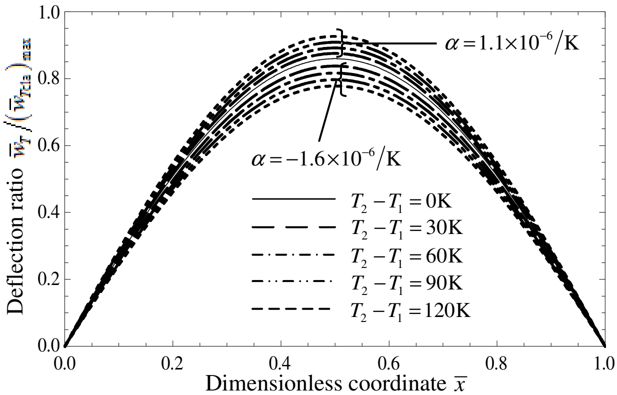

Fig. 3. The effect of nanoscale T on $\bar{w}_T$ / $(\bar{w}_{T\mathrm{cla}})_{\max}$ for SS nanobeam.

deflection of a classical beam shown in Eq. (48). For a nanobeam in high temperature environment, the thermal expansion coefficient and temperature change are taken as $\alpha = 1.1 \times 10^{-6}/\mathrm{K}$ and $T = 100\mathrm{K}$. The nanoscale τ ranges from 0, a classical beam, to 0.2. As observed in Fig. 2, increasing τ tends to reduce the static deflection of the nanobeam. Hence, the classical theory overestimates the thermal-elastic deflection of a shear deformable nanobeam.

In Figure 3 the variation of deflection ratio $\bar{w}_T/(\bar{w}_{T\mathrm{cla}})_{\max}$ along the nanobeam for temperature change ranges from $T = 0\mathrm{K}$ to $T = 120\mathrm{K}$ and $\tau = 0.1$. It is observed that at low and room temperature where $\alpha = -1.6\times10^{-6}$, $\bar{w}_T/(\bar{w}_{T\mathrm{cla}})_{\max}$ decreases as the temperature change increases, while at high temperature where $\alpha = 1.1 \times 10^{-6}$, $\bar{w}_T/(\bar{w}_{T\mathrm{cla}})_{\max}$ increases as the temperature change increases.

The shear deformation effect on dimensionless transverse deflection ratio of nanobeam is clearly demonstrated in Fig. 4. The deflection ratio based on nonlocal

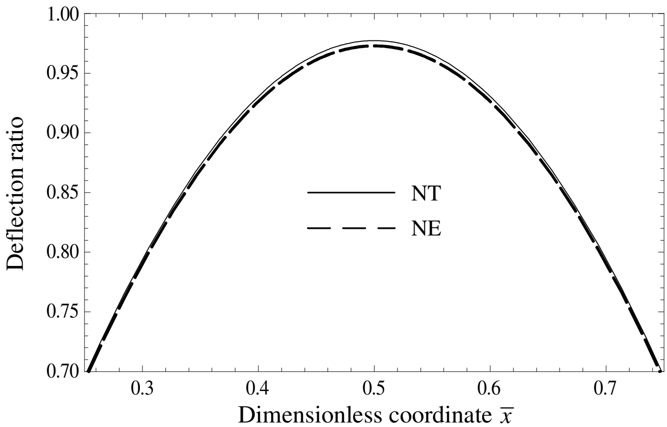

Fig. 4. Transverse deflection ratio using NT and NE model for SS nanobeam.

Timoshenko beam model (NT) and nonlocal Euler beam (NE) is shown in Eq. (42) and Eq. (45) respectively. A nanobeam in high temperature environment, with thermal expansion coefficient $\alpha = 1.1 \times 10^{-6}/\mathrm{K}$ and temperature change $T = 100\mathrm{K}$ is considered. It is observed that the shear deformation effect tends to reduce the nanobeam stiffness where the deflection ratio of the NT model is larger than the corresponding deflection ratio of the NE model.

4.2. *Clamped (CC) Shear Deformable Nanobeam*

For a nanobeam fully clamped in the transverse direction, the six boundary conditions are

$$\bar{w}|_{\bar{x}=0,1} = 0 \quad ; \quad \varphi|_{\bar{x}=0,1} = 0 \quad ; \quad \varphi^{\langle 2 \rangle}\Big|_{\bar{x}=0,1} = 0 \tag{49}$$

Substituting Eqs. (35) and (39) into the boundary conditions above and solving the equations yield six integration constants $C_1, C_2, \ldots, C_6$, and these constants are

$$\begin{aligned} C_1 = \tfrac{\overline{P}\gamma\lambda_2^2}{2\beta\lambda_1 q_1\left(e^{\lambda_1}-1\right)\left(\lambda_1^2-\lambda_2^2\right)} &\left\{2e^{\lambda_1}\left(e^{\lambda_2}+1\right)(\beta\eta-1)\lambda_1\lambda_2\left(\lambda_1^2-\lambda_2^2\right)\right. \\ &+\gamma\lambda_2^3\left(e^{\lambda_1}-1\right)\left(e^{\lambda_2}+1\right)\left[2(\beta\eta-1)-\lambda_1\right] \\ &\left.+\gamma\lambda_1^3(2\beta\eta-\lambda_2-2)\left(e^{\lambda_1}-e^{\lambda_2}-e^{\lambda_1+\lambda_2}+1\right)\right\} \end{aligned} \tag{50}$$

$$\begin{aligned} C_2 = \tfrac{\overline{P}\gamma\lambda_2^2 e^{\lambda_1}}{2\beta\lambda_1 q_1\left(e^{\lambda_1}-1\right)\left(\lambda_1^2-\lambda_2^2\right)} &\left\{2\left(e^{\lambda_2}+1\right)(\beta\eta-1)\lambda_1\lambda_2\left(\lambda_1^2-\lambda_2^2\right)\right. \\ &+\gamma\lambda_1^3(2\beta\eta+\lambda_2-2)\times\left(1-e^{\lambda_1+\lambda_2}\right)+\gamma\lambda_2^3\left(e^{\lambda_1}-1\right)\left(e^{\lambda_2}+1\right) \\ &\left.\left[2(\beta\eta-1)+\lambda_1\right]+\gamma\lambda_1^3(2\beta\eta-\lambda_2-2)\left(e^{\lambda_1}-e^{\lambda_2}\right)\right\} \end{aligned} \tag{51}$$

$$\begin{aligned} C_3 = {} & \tfrac{\overline{P}\gamma\lambda_1^2}{2\beta\lambda_2 q_2\left(e^{\lambda_2}-1\right)\left(\lambda_2^2-\lambda_1^2\right)}\left\{2e^{\lambda_2}\left(e^{\lambda_1}+1\right)(\beta\eta-1)\lambda_1\lambda_2\left(\lambda_2^2-\lambda_1^2\right)\right. \\ & +2\gamma\lambda_2^3\left(e^{\lambda_1}-1\right)\left(e^{\lambda_2}+1\right)(\beta\eta-1) \\ & \left.+\gamma\lambda_1\left(e^{\lambda_1}+1\right)\left(e^{\lambda_2}-1\right)\left[\lambda_2^3-\lambda_1^2\left(2-2\beta\eta+\lambda_2\right)\right]\right\} \end{aligned} \tag{52}$$

$$\begin{aligned} C_4 = {} & \tfrac{\overline{P}\gamma\lambda_1^2 e^{\lambda_2}}{2\beta\lambda_2 q_2\left(e^{\lambda_2}-1\right)\left(\lambda_2^2-\lambda_1^2\right)}\left\{2\left(e^{\lambda_1}+1\right)(\beta\eta-1)\lambda_1\lambda_2\left(\lambda_2^2-\lambda_1^2\right)\right. \\ & +2\gamma\lambda_2^3\left(e^{\lambda_1}-1\right)\left(e^{\lambda_2}+1\right)(\beta\eta-1) \\ & \left.+\gamma\lambda_1\left(e^{\lambda_1}+1\right)\left(e^{\lambda_2}-1\right)\left[\lambda_1^2\left(2\beta\eta+\lambda_2-2\right)-\lambda_2^3\right]\right\} \end{aligned} \tag{53}$$

$$\begin{aligned} C_5 = {} & \tfrac{\overline{P}\gamma}{2\beta q_2}\left\{2\lambda_2^3\left(e^{\lambda_1}-1\right)\left(e^{\lambda_2}+1\right)(\beta\eta-1)+\left(e^{\lambda_1}-1\right)\left(e^{\lambda_2}+1\right)\lambda_1\lambda_2^3\right. \\ & \left.+\lambda_1^3\left(e^{\lambda_1}+1\right)\left(2\beta\eta-\lambda_2-2\right)-\lambda_1^3 e^{\lambda_2}\left(e^{\lambda_1}+1\right)\left(2\beta\eta+\lambda_2-2\right)\right\} \end{aligned} \tag{54}$$

$$\begin{aligned} C_6 = {} & \tfrac{\overline{P}\gamma}{2\beta\lambda_1\lambda_2 q_1\left(e^{\lambda_2}-1\right)\left(e^{\lambda_1}-1\right)\left(\lambda_1^2-\lambda_2^2\right)}\left\{4(\beta\eta-1)\lambda_1\lambda_2\left(\lambda_1^2-\lambda_2^2\right)\right. \\ & \times\left[\lambda_1^3 e^{\lambda_2}\left(e^{2\lambda_1}-1\right)-\lambda_1^3 e^{\lambda_1}\left(e^{2\lambda_2}-1\right)\right] \\ & -\gamma\left[\lambda_1^3\left(e^{\lambda_1}+1\right)\left(e^{\lambda_2}-1\right)-\lambda_2^3\left(e^{\lambda_1}-1\right)\left(e^{\lambda_2}+1\right)\right] \\ & \times\left[-2\lambda_2^3\left(e^{\lambda_1}+1\right)\left(e^{\lambda_2}-1\right)(\beta\eta-1)-\lambda_1\lambda_2^3\left(e^{\lambda_1}-1\right)\left(e^{\lambda_2}-1\right)\right. \\ & \left.\left.+\lambda_1^3\left(e^{\lambda_1}-1\right)\left(2\beta\eta-\lambda_2-2\right)+\lambda_1^3 e^{\lambda_2}\left(e^{\lambda_1}-1\right)\left(2\beta\eta+\lambda_2-2\right)\right]\right\} \end{aligned} \tag{55}$$

where

$$\begin{aligned} q_1 = {} & \left(e^{\lambda_1}+1\right)\left(e^{\lambda_2}+1\right)\lambda_1\lambda_2\left(\lambda_1^2-\lambda_2^2\right)-2\gamma\left(e^{\lambda_1}+1\right)\left(e^{\lambda_2}-1\right)\lambda_1^3 \\ & +2\gamma\left(e^{\lambda_1}-1\right)\left(e^{\lambda_2}+1\right)\lambda_2^3 \end{aligned} \tag{56}$$

$$\begin{aligned} q_2 = {} & \left(e^{\lambda_1}+1\right)\left(e^{\lambda_2}+1\right)\lambda_1\lambda_2\left(\lambda_2^2-\lambda_1^2\right)+2\gamma\left(e^{\lambda_1}+1\right)\left(e^{\lambda_2}-1\right)\lambda_1^3 \\ & -2\gamma\left(e^{\lambda_1}-1\right)\left(e^{\lambda_2}+1\right)\lambda_2^3 \end{aligned} \tag{57}$$

Hence, substituting the above constants $C_1, C_2, \ldots, C_6$ into Eqs. (35) and (39), the thermal-elastic deflection $\bar{w}_T$ and rotation φ_T for a shear deformable CC nanobeam can be obtained. For $GA\kappa_s \to \infty$, $\bar{w}_T$ can be reduced to the thermal deflection $\bar{w}_E$ for a Euler nanobeam, which can be shown as

$$\begin{aligned} \bar{w}_E = {} & \tfrac{\bar{P}}{2\beta}\left[\tfrac{\lambda_1^2\left(e^{\lambda_2\bar{x}}+e^{\lambda_2-\lambda_2\bar{x}}\right)}{\lambda_2\left(e^{\lambda_2}-1\right)\left(\lambda_1^2-\lambda_2^2\right)}-\tfrac{\lambda_2^2\left(e^{\lambda_1\bar{x}}+e^{\lambda_1-\lambda_1\bar{x}}\right)}{\lambda_1\left(e^{\lambda_1}-1\right)\left(\lambda_1^2-\lambda_2^2\right)}\right. \\ & \left.-\tfrac{\lambda_1^3\left(e^{\lambda_1}-1\right)\left(e^{\lambda_2}+1\right)-\lambda_2^3\left(e^{\lambda_1}+1\right)\left(e^{\lambda_2}-1\right)}{\lambda_1\lambda_2\left(e^{\lambda_1}-1\right)\left(e^{\lambda_2}-1\right)\left(\lambda_1^2-\lambda_2^2\right)}\right] \end{aligned} \tag{58}$$

where $\beta = \frac{L^2 P_{xx}}{EI}$. Furthermore, when the nonlocal effect $\tau \to 0$, $\bar{w}_T$ can be reduced to the classical shear deformable deflection $\bar{w}_{T\mathrm{cla}}$, as

$$
\begin{aligned}
\bar{w}_{T\mathrm{cla}} = & \frac{\bar{P}\gamma e^{-\sqrt{\beta}\bar{x}}}{2\beta^{3/2}\left(e^{\sqrt{\beta}}-1\right)\left[\sqrt{\beta}\left(e^{\sqrt{\beta}}+1\right)-2\gamma\left(e^{\sqrt{\beta}}-1\right)\right]} \left\{e^{2\sqrt{\beta}\bar{x}}\gamma\left(2+\sqrt{\beta}-2\beta\eta\right)\right. \\
& -e^{(1+\bar{x})\sqrt{\beta}}\left[\sqrt{\beta}\left(\gamma-2\right)+2\gamma+2\beta^{3/2}\eta-2\beta\gamma\eta\right] \\
& -e^{\sqrt{\beta}}\left[\sqrt{\beta}\left(\gamma-2\right)-2\gamma+2\beta^{3/2}\eta+2\beta\gamma\eta\right] \\
& -2e^{(1+\bar{x})\sqrt{\beta}}\sqrt{\beta}\left[2-\gamma+2\bar{x}^2\gamma-2\beta\eta+2\bar{x}\left(\beta\eta-1\right)\right] \\
& +e^{2\sqrt{\beta}}\gamma\left(\sqrt{\beta}+2\beta\eta-2\right) \\
& +e^{(2+\bar{x})\sqrt{\beta}}\left[\bar{x}^2\left(2\sqrt{\beta}\gamma-\beta\right)+\left(\bar{x}\sqrt{\beta}-\gamma\right)\left(2\beta\eta-2+\sqrt{\beta}\right)\right] \\
& \left.+e^{\sqrt{\beta}\bar{x}}\left[\bar{x}^2\left(\beta+2\sqrt{\beta}\gamma\right)+\left(\bar{x}\sqrt{\beta}+\gamma\right)\left(2\beta\eta-2-\sqrt{\beta}\right)\right]\right\}
\end{aligned}
\tag{59}
$$

The response of dimensionless deflection ratio $\bar{w}_T/\left(\bar{w}_{T\mathrm{cla}}\right)_{\max}$ under high temperature filed with $T = 100\mathrm{K}$ and $\alpha = 1.1\times10^{-6}/\mathrm{K}$ is illustrated in Fig. 5 with τ ranges from 0 to 0.2. The maximum classical shear deformable deflection $\left(\bar{w}_{T\mathrm{cla}}\right)_{\max}$ occur at $\bar{x} = 1/2$. As observed in the figures, increasing τ tends to reduce $\bar{w}_T/\left(\bar{w}_{T\mathrm{cla}}\right)_{\max}$. Hence, the classical theory overestimates the thermal-elastic deflection of a nanobeam.

Plotted in Fig. 6 is the effects of temperature change T on $\bar{w}_T/\left(\bar{w}_{T\mathrm{cla}}\right)_{\max}$ with $\tau = 0.1$ and in low and high temperature environments, where T ranges from 0K to 120K. Similar to the previous example, it is also observed that $\bar{w}_T/\left(\bar{w}_{T\mathrm{cla}}\right)_{\max}$ decreases as T increases for $\alpha < 0$, while $\bar{w}_T/\left(\bar{w}_{T\mathrm{cla}}\right)_{\max}$ increases as T increases for $\alpha > 0$.

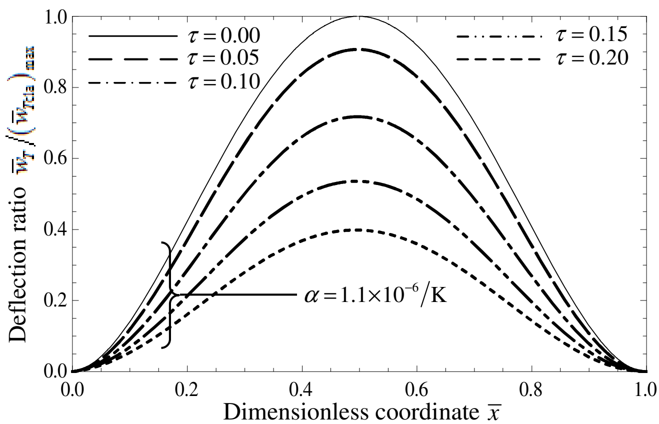

Fig. 5. The effect of nanoscale τ on $\bar{w}_T$ / $(\bar{w}_{T\mathrm{cla}})_{\max}$ for CC nanobeam.

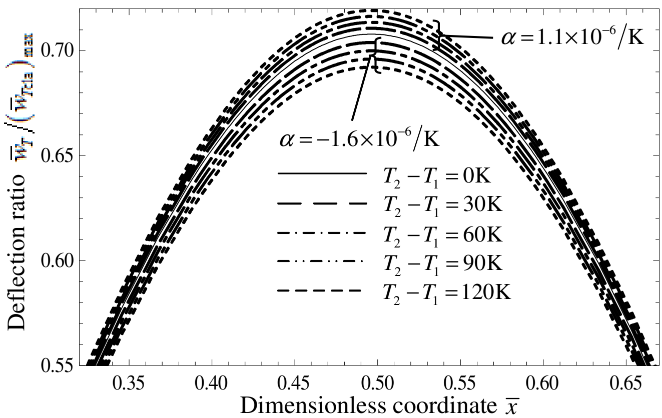

Fig. 6. The effect of nanoscale T on $\bar{w}_T$ / $(\bar{w}_{T\text{cla}})_{\max}$ for CC nanobeam.

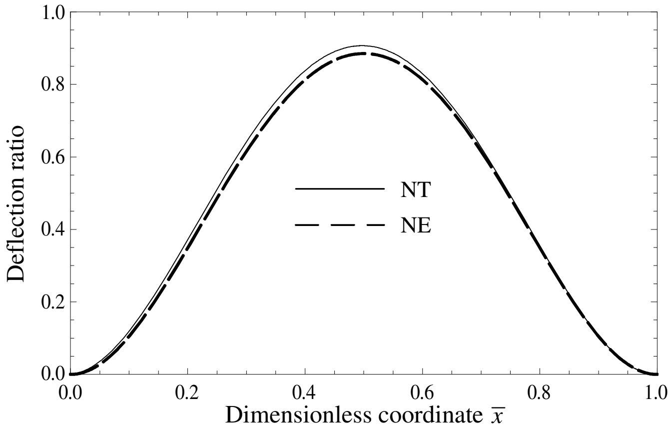

Fig. 7. Transverse deflection ratio using NT and NE model for CC nanobeam.

The comparison between thermal-elastic deflection $\bar{w}_T$ and $\bar{w}_E$, based on Timoshenko beam model (NT) and Euler beam model (NE) respectively, are demonstrated in Fig. 7. It is shown in the figure that shear deformable nanobeam have lager deflection ratio than the one without shear deformation. Therefore, shear deformation effect tends to reduce the stiffness of a nanobeam.

4.3. *Propped Cantilever (CS) Shear Deformable Nanobeam*

For a propped cantilever nanobeam clamped at $\bar{x} = 0$ and simply supported at $\bar{x} = 1$, the six boundary conditions in the transverse direction are

$$\begin{aligned} &\bar{w}\big|_{\bar{x}=0} = 0\ ;\ \varphi\big|_{\bar{x}=0} = 0 \quad ;\ \varphi^{\langle 2\rangle}\big|_{\bar{x}=0} = 0 \\ &\bar{w}\big|_{\bar{x}=1} = 0\ ;\ \varphi^{\langle 1\rangle}\big|_{\bar{x}=1} = 0\ ;\ \varphi^{\langle 3\rangle}\big|_{\bar{x}=1} = 0 \end{aligned} \tag{60}$$

Substituting Eqs. (35) and (39) into the boundary conditions above and solving the equations yield six integration constants $C_1, C_2, \ldots, C_6$ which can be expressed as

$$\begin{aligned} C_1 = \tfrac{\bar{P}\gamma\lambda_2}{2\beta\lambda_1^2\left(\lambda_1^2-\lambda_2^2\right)q_3} &\left\{2e^{\lambda_1}\lambda_1\lambda_2^2\left(\lambda_1^2-\lambda_2^2\right)\left(e^{2\lambda_2}+1\right)(\beta\eta-1)\right. \\ &-2e^{\lambda_1}\lambda_1^3\gamma\lambda_2\left(e^{2\lambda_2}-1\right)(\beta\eta-1) \\ &+2\lambda_2^4\gamma\left(e^{\lambda_1}-1\right)\left(e^{2\lambda_2}+1\right)(\beta\eta-1)-\lambda_1^2\lambda_2^4\gamma\left(e^{2\lambda_2}+1\right) \\ &\left.+\lambda_1^4\gamma\left[2\left(2e^{\lambda_2}-1\right)(\beta\eta-1)+\lambda_2^2+e^{2\lambda_2}\left(2\beta\eta-2+\lambda_2^2\right)\right]\right. \end{aligned} \tag{61}$$

$$\begin{aligned} C_2 = \tfrac{\bar{P}\gamma\lambda_2 e^{\lambda_1}}{2\beta\lambda_1^2\left(\lambda_1^2-\lambda_2^2\right)q_3} &\left\{-\left(e^{\lambda_1}+e^{\lambda_1+2\lambda_2}\right)\gamma\left(\lambda_1^2-\lambda_2^2\right)\right. \\ &\left[2\lambda_2^2(\beta\eta-1)+\lambda_1^2\left(2\beta\eta-2+\lambda_2^2\right)\right] \\ &+4e^{\lambda_1+\lambda_2}\gamma(\beta\eta-1)-2e^{2\lambda_2}\lambda_2(\beta\eta-1)(\lambda_1+\lambda_2) \\ &\left[\lambda_1\lambda_2(\lambda_2-\lambda_1)+\gamma\left(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2\right)\right] \\ &+2\lambda_2(\beta\eta-1)(\lambda_1-\lambda_2)\left[\lambda_1\lambda_2(\lambda_2+\lambda_1)+\gamma\left(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2\right)\right] \end{aligned} \tag{62}$$

$$\begin{aligned} C_3 = \tfrac{\bar{P}\gamma\lambda_1}{2\beta\lambda_2^2 q_3\left(\lambda_1^2-\lambda_2^2\right)} &\left\{2e^{\lambda_2}\lambda_2\lambda_1^2\left(\lambda_2^2-\lambda_1^2\right)\left(e^{2\lambda_1}+1\right)(\beta\eta-1)\right. \\ &-2e^{\lambda_2}\lambda_2^3\gamma\lambda_1\left(e^{2\lambda_1}-1\right)(\beta\eta-1) \\ &+2\lambda_2^4\gamma\left(e^{\lambda_1}-1\right)^2(\beta\eta-1)+\lambda_1^2\lambda_2^4\gamma\left(e^{2\lambda_1}+1\right)+\lambda_1^4\gamma\left(e^{2\lambda_1}+1\right) \\ &\left[2-2\beta\eta+2e^{\lambda_2}(\beta\eta-1)-\lambda_2^2\right] \end{aligned} \tag{63}$$

$$\begin{aligned} C_4 = \tfrac{\bar{P}\gamma\lambda_1 e^{\lambda_2}}{2\beta\lambda_2^2 q_3\left(\lambda_1^2-\lambda_2^2\right)} &\left\{2\lambda_2\lambda_1^2\left(\lambda_2^2-\lambda_1^2\right)\left(e^{2\lambda_1}+1\right)(\beta\eta-1)\right. \\ &-2\lambda_2^3\lambda_1\gamma\left(e^{2\lambda_1}-1\right)(\beta\eta-1) \\ &-2e^{\lambda_2}\lambda_2^4\gamma\left(e^{\lambda_1}-1\right)^2(\beta\eta-1)+\lambda_1^4\gamma\left(e^{2\lambda_1}+1\right) \\ &\left[2-2\beta\eta+e^{\lambda_2}\left(-2+2\beta\eta+\lambda_2^2\right)\right] \end{aligned} \tag{64}$$

$$\begin{aligned} C_5 = \tfrac{\bar{P}\gamma}{2\beta\lambda_1^2\lambda_2^2 q_3} &\left\{-2\,\lambda_2^4\left(e^{\lambda_1}-1\right)^2\left(e^{2\lambda_2}+1\right)(\beta\eta-1)-\lambda_1^2\lambda_2^4\right. \\ &\left(e^{2\lambda_1}+1\right)\left(e^{2\lambda_2}+1\right)+\lambda_1^4\left(e^{2\lambda_1}+1\right) \\ &\left[2\left(2e^{\lambda_2}+1\right)(\beta\eta-1)+\lambda_2^2+e^{2\lambda_2}\left(2\beta\eta-2+\lambda_2^2\right)\right] \end{aligned} \tag{65}$$

$$\begin{aligned} C_6 = \tfrac{\bar{P}\gamma}{2\beta\lambda_1^2\lambda_2^2 q_3} &\left\{2\lambda_1\lambda_2(\beta\eta-1)\left[\left(e^{\lambda_2}+e^{2\lambda_1+\lambda_2}\right)\lambda_1^4-\left(e^{\lambda_1}+e^{\lambda_1+2\lambda_2}\right)\lambda_2^4\right]\right. \\ &-\gamma\left[\lambda_1^3\left(e^{2\lambda_1}+1\right)\left(e^{2\lambda_2}-1\right)-\lambda_2^3\left(e^{2\lambda_1}-1\right)\left(e^{2\lambda_2}+1\right)\right] \\ &\left[2(\beta\eta-1)\left(\lambda_1^2+\lambda_2^2\right)+\lambda_1^2\lambda_2^2\right] \end{aligned} \tag{66}$$

where

$$q_3 = \left(e^{2\lambda_1}+1\right)\left(e^{2\lambda_2}+1\right)\lambda_1\lambda_2\left(\lambda_1^2-\lambda_2^2\right)-\gamma\lambda_1^3\left(e^{2\lambda_1}+1\right)\left(e^{2\lambda_2}-1\right) + \gamma\lambda_2^3\left(e^{2\lambda_1}-1\right)\left(e^{2\lambda_2}+1\right) \tag{67}$$

Hence, substituting $C_1, C_2, \ldots, C_6$ into Eqs. (35) and (39), the thermal-elastic deflection $\bar{w}_T$ and rotation φ_T for shear deformable CS nanobeam can be obtained. For $GA\kappa_s \to \infty$, $\bar{w}_T$ can be reduced to the thermal deflection $\bar{w}_E$ for a Euler nanobeam. Furthermore, for vanishing nonlocal effect $\tau \to 0$, $\bar{w}_T$ can be reduced to the classical shear deformable deflection $\bar{w}_{T\text{cla}}$, as

$$\begin{aligned}\bar{w}_{T\text{cla}} = {} & \frac{\bar{P}\gamma e^{-\sqrt{\beta}\bar{x}}}{2\beta^2\left[\left(e^{2\sqrt{\beta}}+1\right)\sqrt{\beta}-\gamma\left(e^{2\sqrt{\beta}}-1\right)\right]} \\ & \left\{-e^{\sqrt{\beta}\bar{x}}\left[\bar{x}^2\beta\left(\gamma+\sqrt{\beta}\right)-\left(\bar{x}\sqrt{\beta}+\gamma\right)(\beta+2\beta\eta-2)\right]\right. \\ & -2e^{(1+2\bar{x})\sqrt{\beta}}\left(\sqrt{\beta}-\gamma\right)(\beta\eta-1)-2e^{\sqrt{\beta}}\left(\sqrt{\beta}+\gamma\right)(\beta\eta-1) \\ & +\gamma\left(e^{2\sqrt{\beta}}-e^{2\bar{x}\sqrt{\beta}}\right)(\beta+2\beta\eta-2) \\ & +e^{(2+\bar{x})\sqrt{\beta}}\left[\bar{x}^2\beta\left(\gamma-\sqrt{\beta}\right)+\left(\bar{x}\sqrt{\beta}-\gamma\right)(\beta+2\beta\eta-2)\right] \\ & \left.-4e^{(1+\bar{x})\sqrt{\beta}}\sqrt{\beta}\,(\bar{x}-1)(\beta\eta-1)\right\}\end{aligned} \tag{68}$$

Figure 8 illustrates the dimensionless deflection ratio $\bar{w}_T/\left(\bar{w}_{T\text{cla}}\right)_{\max}$ along the nanobeam in a high temperature field with $T = 100\text{K}$ and $\alpha = 1.1 \times 10^{-6}/\text{K}$. The maximum shear deformable deflection $\left(\bar{w}_{T\text{cla}}\right)_{\max}$ occurs at $\bar{x} = \left(15-\sqrt{33}\right)/16$ and the nonlocal effect τ ranges from 0 to 0.2. As observed in the figure, increasing τ tends to reduce $\bar{w}_T/\left(\bar{w}_{T\text{cla}}\right)_{\max}$ in high temperature environment. Hence, the classical theory overestimates the thermal-elastic deflection of shear deformable nanobeam.

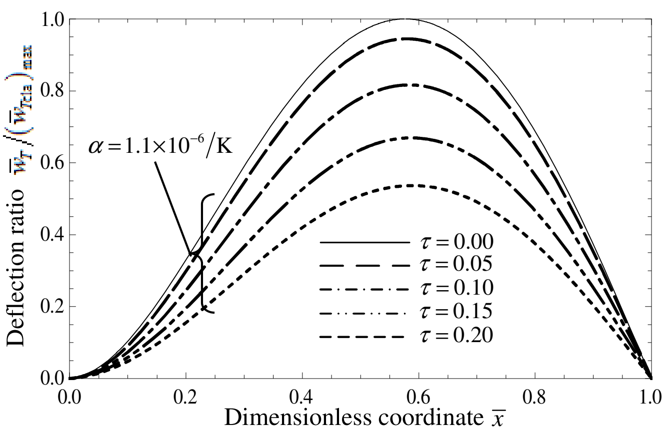

Fig. 8. The effect of nanoscale τ on $\bar{w}_T \,/\, (\bar{w}_{T\text{cla}})_{\max}$ for CS nanobeam.

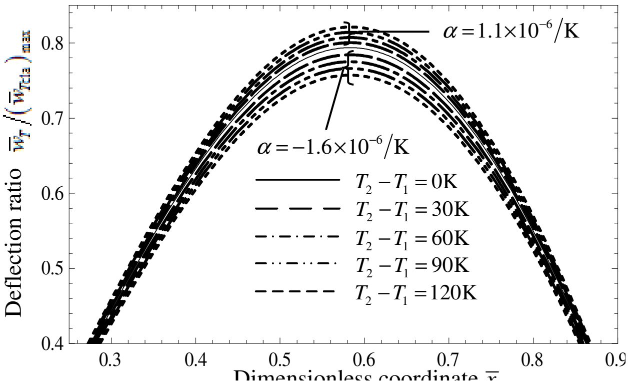

Fig. 9. The effect of nanoscale T on $\bar{w}_T \ / \ (\bar{w}_{T\mathrm{cla}})_{\max}$ for CS nanobeam.

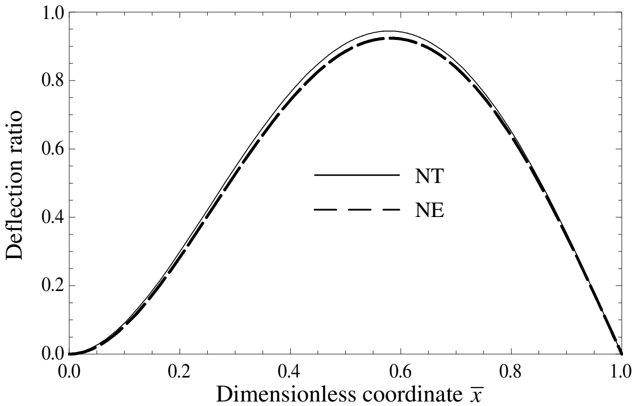

Fig. 10. Transverse deflection ratio using NT and NE model for CS nanobeam.

Figure 9 shows the effect of temperature change T on $\bar{w}_T/\left(\bar{w}_{T\mathrm{cla}}\right)_{\max}$ with $\tau = 0.1$ both in low and high temperature environments. It is also observed that at low and room temperature $\bar{w}_T/\left(\bar{w}_{T\mathrm{cla}}\right)_{\max}$ decreases as the temperature change increases, while at high temperature $\bar{w}_T/\left(\bar{w}_{T\mathrm{cla}}\right)_{\max}$ increases as the temperature change increases. The effect of shear deformation is presented in Fig. 10. It is observed again that shear deformation effect tends to reduce the stiffness of nanobeam where the deflection ratio using the NT model is larger than the corresponding deflection ratio using the NE model.

5. Conclusion

An exact nonlocal stress model for thermal bending of shear deformable nanobeam has been established through the variational principle. New higher-order equilibrium equations and the corresponding boundary conditions for shear deformable nanobeams have been derived. Exact analytical solutions have been presented and the solutions conclude that nonlocal stress effect tends to significantly increase nanobeam stiffness, while shear deformation leads decreasing nanobeam stiffness. It is also concluded that in at low and room temperatures, increasing temperature change causes reduced deflection while in high temperatures, it causes higher deflection. This conclusion is attributed to the positive and negative thermal expansion coefficient, respectively. The formulation, solution methodologies and analytical results presented in this paper will hopefully be helpful for the understanding of mechanical behaviours of MEMS or NEMS systems and devices.

Acknowledgements

This work was supported by a research grant from City University of Hong Kong (Project No. 7002699).

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