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A GENERAL NONLINEAR THIRD-ORDER THEORY OF FUNCTIONALLY GRADED PLATES

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A general third-order plate theory that accounts for geometric nonlinearity and two-constituent material variation through the plate thickness (i.e., functionally graded plates) is presented using the dynamic version of the principle of virtual displacements. The formulation is based on power-law variation of the material through the thickness and the von Kármán nonlinear strains. The governing equations of motion derived herein for a general third-order theory with geometric nonlinearity and material gradation through the thickness are specialized to the existing classical and shear deformation plate theories in the literature. The theoretical developments presented herein can be used to develop finite element models and determine the effect of the geometric nonlinearity and material grading through the thickness on the bending, vibration, and buckling and postbuckling response of elastic plates.

Keywords: General third-order shear deformation plate theory; functionally graded materials; temperature-dependent properties; von Kármán geometric nonlinearity.

1. Background

1.1. *Introduction*

On the threshold of 21st century, we are entering into research and development of the next generation of materials for aerospace structures. The next generation of material systems will feature *electro-thermo-mechanical coupling*, *functionality*, *intelligence*, and *miniaturization* (down to nano length scales). These systems operate under varying conditions — they span the whole spectrum of magneto-electro-thermomechanical conditions. When functionally graded material systems are used in nano- and micro-components, it is necessary to account for size-dependent features, higher-order kinematics, and geometric nonlinearity. The present study is an attempt in developing a general third-order plate theory with higher-order kinematics and geometric nonlinearity. The following subsections provide a brief background for the present study.

1.2. *Functionally graded materials*

Functionally gradient materials (FGMs) are a class of composites that have a gradual variation of material properties from one surface to another; see Hasselman and Youngblood [1978], Yamaouchi, *et al.* [1990] and Koizumi [1993]. These novel materials were proposed as thermal barrier materials for applications in space planes, space structures, nuclear reactors, turbine rotors, flywheels, and gears, to name only a few. As conceived and manufactured today, these materials are isotropic and nonhomogeneous. One reason for increased interest in FGMs is that it may be possible to create certain types of FGM structures capable of adapting to operating conditions.

Two-constituent FGMs are usually made of a mixture of ceramic and metals for use in thermal environments. The ceramic constituent of the material provides the high temperature resistance due to its low thermal conductivity. The ductile metal constituent, on the other hand, prevents fracture due to high temperature gradient in a very short period of time. The gradation in properties of the material reduces thermal stresses, residual stresses, and stress concentration factors.

Noda [1991] presented an extensive review that covers a wide range of topics from thermo-elastic to thermo-inelastic problems. He also discussed the importance of temperature dependent properties on thermoelastics problems. He further presented analytical methods to handle transient heat conduction problems and indicates the presented analytical methods to handle transient heat conduction comprehensive review on the thermoelastic analyses of functionally graded materials.

A number of other investigations dealing with thermal stresses and deformations beams, plates, and cylinders had been published in the literature (see, for example, Noda and Tsuji [1991], Obata, Noda, and Tsuji [1992], Reddy and Chin [1998], Praveen and Reddy [1998], Praveen, Chin, and Reddy [1999], Reddy [2000], Vel and Batra [2002], among others). Among these studies that concern the thermoelastic analysis of plates, beams, or cylinders made of FGMs where the material properties have been considered temperature dependent are dependent are Noda and Tsuji [1991], Praveen and Reddy [1998], Praveen, Chin, and Reddy [1999], Yang and Shen [2003], and Kitipornchai, Yang, and Liew [2004], among several others. The work of Praveen and Reddy [1998], Reddy [2000], and Aliaga and Reddy [2004] also considered von Kármán nonlinearity. The work of Wilson and Reddy was also considered the third-order plate theory of Reddy [1984a,b, 1987, 1990, 2004].

1.3. *Present study*

To the best of the author's knowledge, no work has been reported till date which concerns the thermo-mechanical analysis of FGM plates using a general third-order theory with the von Kármán geometric nonlinearity. This very fact motivates the development of a general third-order theory that contains, as special cases, all existing plate theories. The third-order plate theory accounts for through-thickness power-law variation of a two-constituent material with temperature-dependent material properties and moderate nonlinearity through the von Kármán strains. The present study is a generalization and extension of the third-order theory presented in the works of Reddy [1987, 1990] to study nonlinear deformation of functionally graded plates with temperature dependent properties.

2. Constitutive Models

2.1. *Material variation through the thickness*

Consider a plate made of two-constituent functionally graded material. The x and y coordinates are taken in the midplane of the plate with the z-axis being normal to the plate. Generally the variation of a typical material property of the material in the FGM plate along the thickness coordinate z is assumed to be represented by the simple power-law as (see Praveen and Reddy [1998] and Reddy [2000])

$$P(z,T) = [P_c(T) - P_m(T)]f_n(z) + P_m(T), \quad f_n(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n \tag{1}$$

where P_c and P_m are the material properties of the ceramic and metal faces of the plate, respectively, n is the volume fraction exponent (power-law index). Note that when $n = 0$, we obtain the single-material beam (with property P_c). Figure 1 shows the variation of the volume fraction of ceramic, $f_n(z)$, through the beam thickness for various values of the power-law index n. Note that the volume fraction $f_n(z)$ decreases with increasing value of n.

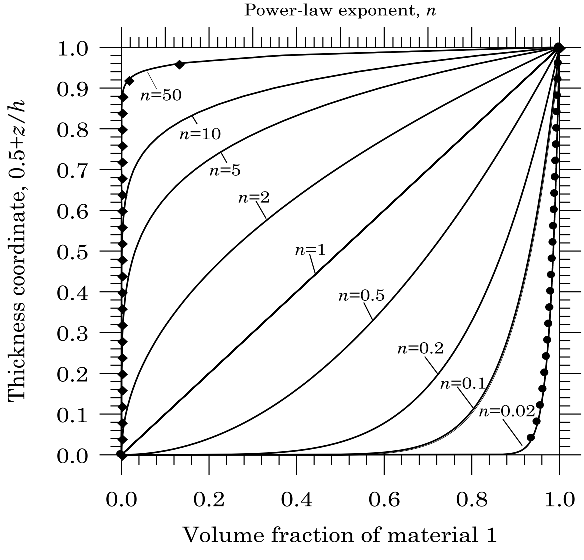

Fig. 1. Volume fraction of ceramic material, $f_n(z)$, through the plate thickness for various values of power-law index.

Since FGMs are generally used in high temperature environment, the material properties are temperature-dependent and they can be expressed as

$$P_\alpha(T) = c_0(c_{-1}T^{-1} + 1 + c_1T + c_2T^2 + c_3T^3), \quad \alpha = c \text{ or } m \tag{2}$$

where c_0 is a constant appearing in the cubic fit of the material property with temperature; and c_{-1}, c_1, c_2, and c_3 coefficients of T^{-1}, T, T^2, and T^3, obtained after factoring out c_0 from the cubic curve fit of the property. The material properties were expressed in this way, so that the higher-order effects of the temperature on the material properties would be readily discernible. For the analysis with constant properties, the material properties were all evaluated at 25.15° C. The values of each of the coefficients appearing in Eq. (2) are listed for the metal and ceramic, from Tables 1 and 2. Also, the material property P at any point along the thickness of the compositionally graded plate is expressed as in Eq. (1). The modulus of elasticity, conductivity, and the coefficient of thermal expansion are considered to vary according to Eqs. (1) and (2).

Table 1. Material properties of Zirconia.

Property	c_0	c_{-1}	$c_1 \times 10^4$	$c_2 \times 10^8$	$c_3 \times 10^{10}$
ρ, Density (kg/m^3)	5,700	0	0	0	0
k, Conductivity (W/m K)	1.7	0	1.276	664.85	0
α, Coefficient of thermal expansion (K)	12.7657×10^{-6}	0	−14.9	0.0001	−0.06775
ν, Poisson's ratio	0.2882	0	1.13345	0	0
C_v, Specific heat (J/kg K)	487.34279	0	3.04908	−6.037232	0
E, Young's modulus (Pa)	244.26596×10^9	0	−13.707	121.393	−3.681378

Table 2. Material properties of Ti6AlV.

Property	c_0	c_{-1}	$c_1 \times 10^4$	$c_2 \times 10^8$	$c_3 \times 10^{10}$
ρ, Density (kg/m^3)	4,429	0	0	0	0
k, Conductivity (W/m K)	1.20947	0	139.375	0	0
α, Coefficient of thermal expansion (K)	7.57876×10^{-6}	0	6.5	31.467	0
ν, Poisson's ratio	0.28838235	0	1.12136	0	0
C_v, Specific heat (J/kg K)	625.29692	0	−4.223876	71.786536	0
E, Young's modulus (Pa)	122.55676×10^9	0	−4.58635	0	−3.681378

3. A General Third-Order Plate Theory

3.1. *Introduction*

Consider a plate of total thickness h and composed of functionally graded material through the thickness. It is assumed that the material is isotropic, and the grading is assumed to be only through the thickness. The $xy-$plane is taken to be the undeformed midplane Ω of the plate with the z-axis positive upward from the midplane, as shown in Fig. 2. We denote the boundary of the midplane with Γ. The plate volume is denoted as $V = \Omega \times (-h/2, h/2)$. The plate is bounded by the top surface Ω^+, bottom surface Ω^-, and the lateral surface $S = \Gamma \times (-h/2, h/2)$.

Here develop a general third-order theory for the deformation of the plate first and then specialize to the well-known plate theories. We restrict the formulation to linear elastic material behavior, small strains, and moderate rotations and displacements, so that there is no geometric update of the domain, that is, the integrals posed on the deformed configuration are evaluated using the undeformed domain and there is no difference between the Cauchy stress tensor and the second Piola–Kirchhoff stress tensor. The equations of motion are obtained using the principle of virtual displacements for the dynamic case (i.e., Hamilton's principle). The three-dimensional problem is reduced to two-dimensional one by assuming a displacement field that is explicit in the thickness coordinate. We make use of the fact that the volume integral of any sufficiently continuous function $f(x, y; z)$ that is explicit in z can be evaluated as

$$\int_V f(x, y; z)\, dV = \int_\Omega \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} f(x, y; z)\, dz \right) dxdy \tag{3}$$

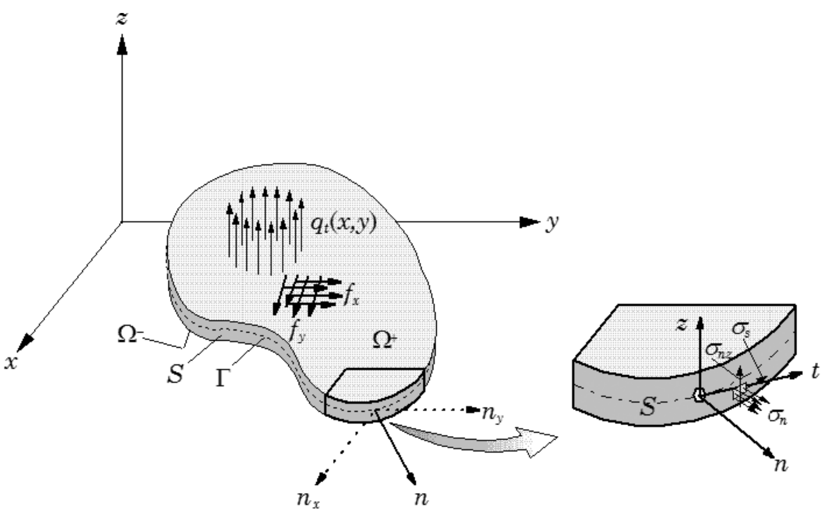

Fig. 2. Domain and various boundary segments of the domain.

When thermal effects are considered, like in the case of thermomechanical loads, the temperature distribution, which is assumed to vary only in the thickness direction, i.e., $T = T(z)$, is determined by first solving a simple steady state heat transfer equation through the thickness of the plate, with specified temperature boundary conditions at the top and bottom of the plate. The energy equation for the temperature variation through the thickness is governed by

$$\rho c_v \frac{\partial T}{\partial t} - \frac{\partial}{\partial z}\left[k(z,T)\frac{\partial T}{\partial z}\right] = 0, \quad -\frac{h}{2} \le z \le \frac{h}{2} \tag{4}$$

$$T(-h/2,t) = T_m(t), \quad T(h/2,t) = T_c(t) \tag{5}$$

where $k(z,T)$ is assumed to vary according to Eqs. (1) and (2), while density ρ and specific heat c_v are assumed to be a constants.

3.2. *Displacements and strains*

We begin with the following displacement field (see Reddy [1987, 1990]):

$$\begin{aligned} u_1(x,y,z,t) &= u(x,y,t) + z\theta_x + z^2\phi_x + z^3\psi_x \\ u_2(x,y,z,t) &= v(x,y,t) + z\theta_y + z^2\phi_y + z^3\psi_y \\ u_3(x,y,z,t) &= w(x,y,t) + z\theta_z + z^2\phi_z \end{aligned} \tag{6}$$

where (u,v,w) are the displacements along the coordinate lines of a material point on the xy-plane, i.e., $u(x,y,t) = u_1(x,y,0,t)$, $v(x,y,t) = u_2(x,y,0,t)$, $w(x,y,t) = u_3(x,y,0,t)$, and

$$\begin{gathered} \theta_x = \left(\frac{\partial u_1}{\partial z}\right)_{z=0}, \quad \theta_y = \left(\frac{\partial u_2}{\partial z}\right)_{z=0}, \quad \theta_z = \left(\frac{\partial u_3}{\partial z}\right)_{z=0} \\ 2\phi_x = \left(\frac{\partial^2 u_1}{\partial z^2}\right)_{z=0}, \quad 2\phi_y = \left(\frac{\partial^2 u_2}{\partial z^2}\right)_{z=0}, \quad 2\phi_z = \frac{\partial^2 u_3}{\partial z^2} \\ 6\psi_x = \frac{\partial^3 u_1}{\partial z^3}, \qquad 6\psi_y = \frac{\partial^3 u_2}{\partial z^3} \end{gathered} \tag{7}$$

The reason for expanding the inplane displacements up to the cubic term and the transverse displacement up to the quadratic term in z is to obtain a quadratic variation of the transverse shear strains $\gamma_{xz} = 2\varepsilon_{xz}$ and $\gamma_{yz} = 2\varepsilon_{yz}$ through the plate thickness. Note that all three displacements contribute to the quadratic variation. In the most general case represented by the displacement field (6), there are 11 generalized displacements $(u,v,w,\theta_x,\theta_y,\theta_z,\phi_x,\phi_y,\phi_z,\psi_x,\psi_y)$ and, therefore, 11 differential equations will be required to determine them.

The nonzero von Kármán type strains associated with the displacement field in Eq. (6) are given by

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + z^2 \begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \gamma_{xy}^{(2)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} \tag{8}$$

$$\begin{Bmatrix} \varepsilon_{zz} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{zz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{zz}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{yz}^{(1)} \end{Bmatrix} + z^2 \begin{Bmatrix} \varepsilon_{zz}^{(2)} \\ \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(2)} \end{Bmatrix} \tag{9}$$

with

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \end{Bmatrix} \tag{10}$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \gamma_{xy}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \psi_x}{\partial x} \\ \frac{\partial \psi_y}{\partial y} \\ \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \end{Bmatrix} \tag{11}$$

$$\begin{Bmatrix} \varepsilon_{zz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \theta_z \\ \theta_x + \frac{\partial w}{\partial x} \\ \theta_y + \frac{\partial w}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{zz}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{yz}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 2\phi_z \\ 2\phi_x + \frac{\partial \theta_z}{\partial x} \\ 2\phi_y + \frac{\partial \theta_z}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{zz}^{(2)} \\ \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3\psi_x + \frac{\partial \phi_z}{\partial x} \\ 3\psi_y + \frac{\partial \phi_z}{\partial y} \end{Bmatrix} \tag{12}$$

where $(\varepsilon_{xx}^{(0)}, \varepsilon_{yy}^{(0)}, \gamma_{xy}^{(0)})$ are the *membrane strains*, $(\varepsilon_{xx}^{(1)}, \varepsilon_{yy}^{(1)}, \gamma_{xy}^{(1)})$ are the flexural (bending) strains, $(\varepsilon_{xx}^{(2)}, \varepsilon_{yy}^{(2)}, \gamma_{xy}^{(2)})$ and $(\varepsilon_{xx}^{(3)}, \varepsilon_{yy}^{(3)}, \gamma_{xy}^{(3)})$ higher-order strains. In writing the strain-displacement relations, we have assumed that the strains are small and rotations are moderately large so that we have

$$\begin{gathered} \left(\frac{\partial u_\alpha}{\partial x}\right)^2 \approx 0, \quad \left(\frac{\partial u_\alpha}{\partial y}\right)^2 \approx 0 \\ \left(\frac{\partial u_3}{\partial x}\right)^2 \approx \left(\frac{\partial w}{\partial x}\right)^2, \quad \left(\frac{\partial u_3}{\partial y}\right)^2 \approx \left(\frac{\partial w}{\partial y}\right)^2 \quad \left(\frac{\partial u_3}{\partial x}\right)\left(\frac{\partial u_3}{\partial y}\right) \approx \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{gathered} \tag{13}$$

If the transverse shear stresses, σ_{xz} and σ_{yz} are required to be zero on the top and bottom of the plate, i.e., $z = \pm h/2$, as in the Reddy third-order theory Reddy [1984a,b, 2004], it is necessary that γ_{xz} and γ_{yz} be zero at $z = \pm h/2$. This in turn yields

$$\begin{aligned}
&2\phi_x + \frac{\partial \theta_z}{\partial x} = 0, \qquad \theta_x + \frac{\partial w}{\partial x} + \frac{h^2}{4}\left(3\psi_x + \frac{\partial \phi_z}{\partial x}\right) = 0\\
&2\phi_y + \frac{\partial \theta_z}{\partial y} = 0, \qquad \theta_y + \frac{\partial w}{\partial y} + \frac{h^2}{4}\left(3\psi_y + \frac{\partial \phi_z}{\partial y}\right) = 0
\end{aligned} \tag{14}$$

Thus, the variables $(\phi_x, \phi_y, \psi_x, \psi_y)$ can be expressed in terms of $(w, \theta_x, \theta_y, \theta_z, \phi_z)$, and thus reduce the number of generalized displacements from 11 to 7. In addition, if we set $\theta_z = \phi_z = 0$, we obtain the displacement field of the Reddy third-order theory, which has only 5 variables $(u, v, w, \theta_x, \theta_y)$.

3.3. *Equations of motion*

The equations of motion can be derived using the principle of virtual displacements. In the derivation, we account for thermal effect with the understanding that the material properties are given functions of temperature, and that the temperature change ΔT is a known function of position from the solution of Eqs. (4) and (5). Thus, temperature field enters the formulation only through constitutive equations.

The principle of virtual displacements for the dynamic case requires that

$$\int_0^T (\delta\mathcal{K} - \delta\mathcal{U} - \delta\mathcal{V})\, dt = 0 \tag{15}$$

where $\delta\mathcal{K}$ is the virtual kinetic energy, $\delta\mathcal{U}$ is the virtual strain energy, and $\delta\mathcal{V}$ is the virtual work done by external forces. Each of these quantities are derived next.

The virtual kinetic energy $\delta\mathcal{K}$ is

$$\begin{aligned}
\delta\mathcal{K} &= \int_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left(\frac{\partial u_1}{\partial t}\frac{\partial \delta u_1}{\partial t} + \frac{\partial u_2}{\partial t}\frac{\partial \delta u_2}{\partial t} + \frac{\partial u_3}{\partial t}\frac{\partial \delta u_3}{\partial t}\right) dz\ dxdy\\
&= \int_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \Big[\left(\dot{u} + z\dot{\theta}_x + z^2\dot{\phi}_x + z^3\dot{\psi}_x\right)\left(\delta\dot{u} + z\delta\dot{\theta}_x + z^2\delta\dot{\phi}_x + z^3\delta\dot{\psi}_x\right)\\
&\quad + \left(\dot{v} + z\dot{\theta}_y + z^2\dot{\phi}_y + z^3\dot{\psi}_y\right)\left(\delta\dot{v} + z\delta\dot{\theta}_y + z^2\delta\dot{\phi}_y + z^3\delta\dot{\psi}_y\right)\\
&\quad + \left(\dot{w} + z\dot{\theta}_z + z^2\dot{\phi}_z\right)\left(\delta\dot{w} + z\delta\dot{\theta}_z + z^2\delta\dot{\phi}_z\right)\Big] dz\, dxdy\\
&= \int_\Omega \big[\big(m_0\dot{u} + m_1\dot{\theta}_x + m_2\dot{\phi}_x + m_3\dot{\psi}_x\big)\delta\dot{u} + \big(m_1\dot{u} + m_2\dot{\theta}_x + m_3\dot{\phi}_x + m_4\dot{\psi}_x\big)\delta\dot{\theta}_x\\
&\quad + \big(m_2\dot{u} + m_3\dot{\theta}_x + m_4\dot{\phi}_x + m_5\dot{\psi}_x\big)\delta\dot{\phi}_x + \big(m_3\dot{u} + m_4\dot{\theta}_x + m_5\dot{\phi}_x + m_6\dot{\psi}_x\big)\delta\dot{\psi}_x\\
&\quad + \big(m_0\dot{v} + m_1\dot{\theta}_y + m_2\dot{\phi}_y + m_3\dot{\psi}_y\big)\delta\dot{v} + \big(m_1\dot{v} + m_2\dot{\theta}_y + m_3\dot{\phi}_y + m_4\dot{\psi}_y\big)\delta\dot{\theta}_y\\
&\quad + \big(m_2\dot{v} + m_3\dot{\theta}_y + m_4\dot{\phi}_y + m_5\dot{\psi}_y\big)\delta\dot{\phi}_y + \big(m_3\dot{v} + m_4\dot{\theta}_y + m_5\dot{\phi}_y + m_6\dot{\psi}_y\big)\delta\dot{\psi}_y\\
&\quad + \big(m_0\dot{w} + m_1\dot{\theta}_z + m_2\dot{\phi}_z\big)\delta\dot{w} + \big(m_1\dot{w} + m_2\dot{\theta}_z + m_3\dot{\phi}_z\big)\delta\dot{\theta}_z\\
&\quad + \big(m_2\dot{w} + m_3\dot{\theta}_z + m_4\dot{\phi}_z\big)\delta\dot{\phi}_z\big] dxdy
\end{aligned} \tag{16}$$

where the superposed dot on a variable indicates time derivative, e.g., $\dot{u} = \partial u/\partial t$, and m_i $(i = 0, 1, 2, \ldots, 6)$ are the mass moments of inertia

$$m_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i \rho \, dz \tag{17}$$

The virtual strain energy is given by

$$\begin{aligned}
\delta\mathcal{U} &= \int_\Omega \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz} + \sigma_{xy}\delta\gamma_{xy} + \sigma_{xz}\delta\gamma_{xz} + \sigma_{yz}\delta\gamma_{yz}) \, dzdxdy \\
&= \int_\Omega \Bigg\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \Bigg[\sigma_{xx} \left(\sum_{i=0}^{3} (z)^i \delta\varepsilon_{xx}^{(i)} \right) + \sigma_{yy} \left(\sum_{i=0}^{3} (z)^i \delta\varepsilon_{yy}^{(i)} \right) + \sigma_{xy} \left(\sum_{i=0}^{3} (z)^i \delta\gamma_{xy}^{(i)} \right) \\
&\quad + \sigma_{zz} \left(\sum_{i=0}^{2} (z)^i \delta\varepsilon_{zz}^{(i)} \right) + \sigma_{xz} \left(\sum_{i=0}^{2} (z)^i \delta\gamma_{xz}^{(i)} \right) + \sigma_{yz} \left(\sum_{i=0}^{2} (z)^i \delta\gamma_{yz}^{(i)} \right) \Bigg] dz \Bigg\} dxdy
\end{aligned} \tag{18}$$

Next, we introduce thickness-integrated stress resultants

$$M_{ij}^{(k)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k \sigma_{ij} dz, \quad (k = 0, 1, 2, 3) \tag{19}$$

Then the virtual strain energy can be expressed in terms of the stress resultants as

$$\begin{aligned}
\delta\mathcal{U} = \int_\Omega \Bigg[&\sum_{i=0}^{3} \left(M_{xx}^{(i)} \delta\varepsilon_{xx}^{(i)} + M_{yy}^{(i)} \delta\varepsilon_{yy}^{(i)} + M_{xy}^{(i)} \delta\gamma_{xy}^{(i)} \right) \\
&+ \sum_{i=0}^{2} \left(M_{zz}^{(i)} \delta\varepsilon_{zz}^{(i)} + M_{xz}^{(i)} \delta\gamma_{xz}^{(i)} + M_{yz}^{(i)} \delta\gamma_{yz}^{(i)} \right) \Bigg] dxdy
\end{aligned} \tag{20}$$

Note that $M_{xx}^{(0)}$, $M_{yy}^{(0)}$, and $M_{xy}^{(0)}$ are the membrane forces (often denoted by N_{xx}, N_{yy}, and N_{xy}), $M_{xx}^{(1)}$, $M_{yy}^{(1)}$, and $M_{xy}^{(1)}$ are the bending moments (denoted by M_{xx}, M_{yy}, and M_{xy}), and $M_{xz}^{(0)}$ and $M_{yz}^{(0)}$ are the shear forces (denoted by Q_x and Q_y). The rest of the stress resultants are higher order generalized forces, which are often difficult to physically interpret. Because of their similarity to the generalized physical forces identified above, they are assumed to be zero when their counter parts (i.e., generalized displacements) are not specified.

The virtual work done by external forces consists of three parts: (1) virtual work done by the body forces in $V = \Omega \times (-h/2, h/2)$, (2) virtual work done by surface tractions acting on the top and bottom surfaces of the plate Ω^+ and Ω^-, and (3) virtual work done by the surface tractions on the lateral surface $S = \Gamma \times (-h/2, h/2)$, where Ω^+ denotes the top surface of the plate, Ω the middle surface of the plate, Ω^- the bottom surface of the plate, and Γ is the boundary of the middle surface (see Fig. 1).

Let $(\bar{f}_x, \bar{f}_y, \bar{f}_z)$ denote the body forces (measured per unit volume), $(\bar{t}_x, \bar{t}_y, \bar{t}_z)$ denote the surface forces (measured per unit area) on S, (q_x^t, q_y^t, q_z^t) denote the forces (measured per unit area) on Ω^+, and (q_x^b, q_y^b, q_z^b) denote the forces (measured per unit area) on Ω^- in the (x, y, z) coordinate directions. Then the virtual work done by external forces is

$$\delta\mathcal{V} = -\left[\int_V \left(\bar{f}_x \delta u_1 + \bar{f}_y \delta u_2 + \bar{f}_z \delta u_3\right) dV + \int_S \left(\bar{t}_x \delta u_1 + \bar{t}_y \delta u_2 + \bar{t}_z \delta u_3\right) dS \right.$$
$$\left. + \int_{\Omega^+} \left(q_x^t \delta u_1 + q_y^t \delta u_2 + q_z^t \delta u_3\right) dxdy + \int_{\Omega^-} \left(q_x^b \delta u_1 + q_y^b \delta u_2 + q_z^b \delta u_3\right) dxdy\right] \tag{21}$$

In view of the displacement field (6), $\delta\mathcal{V}$ can be expressed as

$$\delta\mathcal{V} = -\left\{\int_\Omega \left(f_x^{(0)}\delta u + f_x^{(1)}\delta\theta_x + f_x^{(2)}\delta\phi_x + f_x^{(3)}\delta\psi_x + f_y^{(0)}\delta v + f_y^{(1)}\delta\theta_y + f_y^{(2)}\delta\phi_y \right.\right.$$
$$\left. + f_y^{(3)}\delta\psi_y + f_z^{(0)}\delta w + f_z^{(1)}\delta\theta_z + f_z^{(2)}\delta\phi_z\right) dxdy$$
$$+ \int_\Gamma \left(t_x^{(0)}\delta u + t_x^{(1)}\delta\theta_x + t_x^{(2)}\delta\phi_x + t_x^{(3)}\delta\psi_x + t_y^{(0)}\delta v + t_y^{(1)}\delta\theta_y + t_y^{(2)}\delta\phi_y \right.$$
$$\left. + t_y^{(3)}\delta\psi_y + t_z^{(0)}\delta w + t_z^{(1)}\delta\theta_z + t_z^{(2)}\delta\phi_z\right) d\Gamma$$
$$+ \int_\Omega \left[(q_x^t + q_x^b)\delta u + (q_y^t + q_y^b)\delta v + (q_z^t + q_z^b)\delta w + \frac{h}{2}(q_x^t - q_x^b)\delta\theta_x \right.$$
$$+ \frac{h}{2}(q_y^t - q_y^b)\delta\theta_y + \frac{h}{2}(q_z^t - q_z^b)\delta\theta_z + \frac{h^2}{4}(q_x^t + q_x^b)\delta\phi_x + \frac{h^2}{4}(q_y^t + q_y^b)\delta\phi_y$$
$$\left.\left. + \frac{h^2}{4}(q_z^t + q_z^b)\delta\phi_z + \frac{h^3}{8}(q_x^t - q_x^b)\delta\psi_x + \frac{h^3}{8}(q_y^t - q_y^b)\delta\psi_y\right] dxdy\right\} \tag{22}$$

where

$$f_\xi^{(i)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i \,\bar{f}_\xi\, dz, \quad t_\xi^{(i)} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^i \,\bar{t}_\xi\, dz \tag{23}$$

The equations of motion of the general third-order plate theory governing functionally graded plates accounting for modified couple stresses are obtained by substituting $\delta\mathcal{K}$, $\delta\mathcal{U}$, and $\delta\mathcal{V}$, from Eqs. (16), (20), and (22), respectively, into Eq. (15), applying the integration-by-parts to relieve all virtual generalized displacements of any differentiation, and invoking the fundamental lemma of the variational calculus (see Reddy [2002, 2004]). We obtain (after a lengthy algebra and manipulations) the following equations:

$$\delta u: \quad \frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} + q_x = m_0\ddot{u} + m_1\ddot{\theta}_x + m_2\ddot{\phi}_x + m_3\ddot{\psi}_x \tag{24}$$

$$\delta v: \quad \frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} + q_y = m_0\ddot{v} + m_1\ddot{\theta}_y + m_2\ddot{\phi}_y + m_3\ddot{\psi}_y \tag{25}$$

$$\delta w: \quad \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}M_{xx}^{(0)} + \frac{\partial w}{\partial y}M_{xy}^{(0)}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}M_{xy}^{(0)} + \frac{\partial w}{\partial y}M_{yy}^{(0)}\right)$$

$$+\frac{\partial M_{xz}^{(0)}}{\partial x}+\frac{\partial M_{yz}^{(0)}}{\partial y}+q_z=m_0\ddot{w}+m_1\ddot{\theta}_z+m_2\ddot{\phi}_z \tag{26}$$

$$\delta\theta_x: \quad \frac{\partial M_{xx}^{(1)}}{\partial x}+\frac{\partial M_{xy}^{(1)}}{\partial y}-M_{xz}^{(0)}+F_x=m_1\ddot{u}+m_2\ddot{\theta}_x+m_3\ddot{\phi}_x+m_4\ddot{\psi}_x \tag{27}$$

$$\delta\theta_y: \quad \frac{\partial M_{xy}^{(1)}}{\partial x}+\frac{\partial M_{yy}^{(1)}}{\partial y}-M_{yz}^{(0)}+F_y=m_1\ddot{v}+m_2\ddot{\theta}_y+m_3\ddot{\phi}_y+m_4\ddot{\psi}_y \tag{28}$$

$$\delta\phi_x: \quad \frac{\partial M_{xx}^{(2)}}{\partial x}+\frac{\partial M_{xy}^{(2)}}{\partial y}-2M_{xz}^{(1)}+G_x=m_2\ddot{u}+m_3\ddot{\theta}_x+m_4\ddot{\phi}_x+m_5\ddot{\psi}_x \tag{29}$$

$$\delta\phi_y: \quad \frac{\partial M_{xy}^{(2)}}{\partial x}+\frac{\partial M_{yy}^{(2)}}{\partial y}-2M_{yz}^{(1)}+G_y=m_2\ddot{v}+m_3\ddot{\theta}_y+m_4\ddot{\phi}_y+m_5\ddot{\psi}_y \tag{30}$$

$$\delta\psi_x: \quad \frac{\partial M_{xx}^{(3)}}{\partial x}+\frac{\partial M_{xy}^{(3)}}{\partial y}-3M_{xz}^{(2)}+H_x=m_3\ddot{u}+m_4\ddot{\theta}_x+m_5\ddot{\phi}_x+m_6\ddot{\psi}_x \tag{31}$$

$$\delta\psi_y: \quad \frac{\partial M_{xy}^{(3)}}{\partial x}+\frac{\partial M_{yy}^{(3)}}{\partial y}-3M_{yz}^{(2)}+H_y=m_3\ddot{v}+m_4\ddot{\theta}_y+m_5\ddot{\phi}_y+m_6\ddot{\psi}_y \tag{32}$$

$$\delta\theta_z: \quad \frac{\partial M_{xz}^{(1)}}{\partial x}+\frac{\partial M_{yz}^{(1)}}{\partial y}-M_{zz}^{(0)}+P=m_1\ddot{w}+m_2\ddot{\theta}_z+m_3\ddot{\phi}_z \tag{33}$$

$$\delta\phi_z: \quad \frac{\partial M_{xz}^{(2)}}{\partial x}+\frac{\partial M_{yz}^{(2)}}{\partial y}-2M_{zz}^{(1)}+Q=m_2\ddot{w}+m_3\ddot{\theta}_z+m_4\ddot{\phi}_z \tag{34}$$

where

$$\begin{aligned}
q_x &= f_x^{(0)}+q_x^t+q_x^b, \quad q_y=f_y^{(0)}+q_y^t+q_y^b, \quad q_z=f_z^{(0)}+q_z^t+q_z^b\\
F_x &= f_x^{(1)}+\frac{h}{2}(q_x^t-q_x^b), \quad F_y=f_y^{(1)}+\frac{h}{2}(q_y^t-q_y^b)\\
G_x &= f_x^{(2)}+\frac{h^2}{4}(q_x^t+q_x^b), \quad G_y=f_y^{(2)}+\frac{h^2}{4}(q_y^t+q_y^b)\\
H_x &= f_x^{(3)}+\frac{h^3}{8}(q_x^t-q_x^b), \quad H_y=f_y^{(3)}+\frac{h^3}{8}(q_y^t-q_y^b)\\
P &= f_z^{(1)}+\frac{h}{2}(q_z^t-q_z^b), \quad Q=f_z^{(2)}+\frac{h^2}{4}(q_z^t+q_z^b)
\end{aligned} \tag{35}$$

The boundary conditions involve specifying the following generalized forces that are dual to the generalized displacement $(u,v,w,\theta_x,\theta_y,\phi_x,\phi_y,\psi_x,\psi_y,\theta_z,\phi_z)$:

$$\delta u: \quad \bar{M}_{nn}^{(0)}\equiv M_{xx}^{(0)}n_x+M_{xy}^{(0)}n_y \tag{36}$$

$$\delta v: \quad \bar{M}_{nt}^{(0)}\equiv M_{xy}^{(0)}n_x+M_{yy}^{(0)}n_y \tag{37}$$

$$\begin{aligned}
\delta w: \quad \bar{M}_{nz}^{(0)} &\equiv \left(\frac{\partial w}{\partial x}M_{xx}^{(0)}+\frac{\partial w}{\partial y}M_{xy}^{(0)}\right)n_x+\left(\frac{\partial w}{\partial x}M_{xy}^{(0)}+\frac{\partial w}{\partial y}M_{yy}^{(0)}\right)n_y\\
&\quad +M_{xz}^{(0)}n_x+M_{yz}^{(0)}n_y
\end{aligned} \tag{38}$$

$$\delta\theta_x: \quad \bar{M}_{nn}^{(1)} \equiv M_{xx}^{(1)} n_x + M_{xy}^{(1)} n_y \tag{39}$$

$$\delta\theta_y: \quad \bar{M}_{nt}^{(1)} \equiv M_{xy}^{(1)} n_x + M_{yy}^{(1)} n_y \tag{40}$$

$$\delta\phi_x: \quad \bar{M}_{nn}^{(2)} \equiv M_{xx}^{(2)} n_x + M_{xy}^{(2)} n_y \tag{41}$$

$$\delta\phi_y: \quad \bar{M}_{nt}^{(2)} \equiv M_{xy}^{(2)} n_x + M_{yy}^{(2)} n_y \tag{42}$$

$$\delta\psi_x: \quad \bar{M}_{nn}^{(3)} \equiv M_{xx}^{(3)} n_x + M_{xy}^{(3)} n_y \tag{43}$$

$$\delta\psi_y: \quad \bar{M}_{nt}^{(3)} \equiv M_{xy}^{(3)} n_x + M_{yy}^{(3)} n_y \tag{44}$$

$$\delta\theta_z: \quad \bar{M}_{nz}^{(1)} \equiv M_{xz}^{(1)} n_x + M_{yz}^{(1)} n_y \tag{45}$$

$$\delta\phi_z: \quad \bar{M}_{nz}^{(2)} \equiv M_{xz}^{(2)} n_x + M_{yz}^{(2)} n_y \tag{46}$$

3.4. *Constitutive relations*

Here we represent the profile for volume fraction variation by the expression in Eq. (1); we assume that moduli E and G, density ρ, and thermal coefficient of expansion α vary according to Eq. (1), and ν is assumed to be a constant. The linear constitutive relations are (see Reddy [2008])

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \Lambda \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{12} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{22} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} - \frac{E\alpha\Delta T}{1-2\nu} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{47}$$

where

$$c_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \quad c_{12} = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad c_{22} = \frac{E}{2(1+\nu)}$$

α is the coefficients of thermal expansion, and ΔT is the temperature increment from a reference temperature T_0, $\Delta T = T - T_0$.

3.5. *Plate constitutive equations*

Here we relate the generalized forces ($M_{xx}^{(i)}$, $M_{yy}^{(i)}$, $M_{xy}^{(i)}$) to the generalized displacements $(u, v, w, \theta_x, \theta_y, \phi_x, \phi_y, \psi_x, \psi_y, \theta_z, \phi_z)$. We have

$$\begin{Bmatrix} M_{xx}^{(i)} \\ M_{yy}^{(i)} \\ M_{zz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} (z)^i dz = \sum_{k=i}^{3+i} \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} & A_{12}^{(k)} \\ A_{12}^{(k)} & A_{11}^{(k)} & A_{12}^{(k)} \\ A_{12}^{(k)} & A_{12}^{(k)} & A_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(k-i)} \\ \varepsilon_{yy}^{(k-i)} \\ \varepsilon_{zz}^{(k-i)} \end{Bmatrix} - \begin{Bmatrix} X_T^{(i)} \\ Y_T^{(i)} \\ Z_T^{(i)} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{xy}^{(i)} \\ M_{xz}^{(i)} \\ M_{yz}^{(i)} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} (z)^i dz = \sum_{k=i}^{3+i} \begin{bmatrix} B_{11}^{(k)} & 0 & 0 \\ 0 & B_{11}^{(k)} & 0 \\ 0 & 0 & B_{11}^{(k)} \end{bmatrix} \begin{Bmatrix} \gamma_{xy}^{(k-i)} \\ \gamma_{xz}^{(k-i)} \\ \gamma_{yz}^{(k-i)} \end{Bmatrix} \tag{48}$$

where

$$\begin{aligned} A_{11}^{(k)} &= \frac{\eta}{(1+\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z,T)\, dz, \quad A_{12}^{(k)} = \frac{\zeta}{(1+\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z,T)\, dz \\ B_{11}^{(k)} &= \frac{1}{2(1+\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z,T)\, dz, \quad \eta = \frac{(1-\nu)}{(1-2\nu)}, \quad \zeta = \frac{\nu}{(1-2\nu)} \end{aligned} \tag{49}$$

and the thermal generalized forces are $X_T^{(i)}$, $Y_T^{(i)}$, and $Z_T^{(i)}$ are defined by

$$X_T^{(i)} = Y_T^{(i)} = Z_T^{(i)} = \frac{1}{(1-2\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z,T)\alpha(z,T)\Delta T\, dz \tag{50}$$

We note that $\varepsilon_{zz}^{(3)} = 0$, $\gamma_{xz}^{(3)} = 0$, and $\gamma_{xz}^{(3)} = 0$.

4. Specialization to Other Theories

The general third-order theory developed herein contains all of the existing plate theories and some new theories. They are summarized here.

4.1. *A general third-order theory with traction-free top and bottom surfaces*

If the top and bottom surfaces of the plate are free of any tangential forces, we can invoke the conditions in Eq. (14) and eliminate ϕ_x, ϕ_y, ψ_x, and ψ_y:

$$\begin{aligned} \phi_x &= -\frac{1}{2}\frac{\partial \theta_z}{\partial x}, \quad \psi_x = -\frac{1}{3}\frac{\partial \phi_z}{\partial x} - c_1\left(\theta_x + \frac{\partial w}{\partial x}\right), \quad c_1 = \frac{4}{3h^2} \\ \phi_y &= -\frac{1}{2}\frac{\partial \theta_z}{\partial y}, \quad \psi_y = -\frac{1}{3}\frac{\partial \phi_z}{\partial y} - c_1\left(\theta_y + \frac{\partial w}{\partial y}\right) \end{aligned} \tag{51}$$

Then the higher-order strains take the form

$$\begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \gamma_{xy}^{(2)} \end{Bmatrix} = -\frac{1}{2} \begin{Bmatrix} \frac{\partial^2 \theta_z}{\partial x^2} \\ \frac{\partial^2 \theta_z}{\partial y^2} \\ 2\frac{\partial^2 \theta_z}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = -\frac{1}{3} \begin{Bmatrix} \frac{\partial^2 \phi_z}{\partial x^2} \\ \frac{\partial^2 \phi_z}{\partial y^2} \\ 2\frac{\partial^2 \phi_z}{\partial x \partial y} \end{Bmatrix} - c_1 \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial \theta_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{52}$$

$$\begin{Bmatrix} \varepsilon_{zz}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{yz}^{(1)} \end{Bmatrix} = \begin{Bmatrix} 2\phi_z \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{zz}^{(2)} \\ \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(2)} \end{Bmatrix} = -c_2 \begin{Bmatrix} 0 \\ \theta_x + \frac{\partial w}{\partial x} \\ \theta_y + \frac{\partial w}{\partial y} \end{Bmatrix}, \quad c_2 = \frac{4}{h^2} \tag{53}$$

Then the equations of motion become

$$\delta u: \quad \frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} + q_x = m_0 \ddot{u} + m_1 \ddot{\theta}_x - \frac{m_2}{2}\frac{\partial \ddot{\theta}_z}{\partial x} - m_3\left[c_1\left(\ddot{\theta}_x + \frac{\partial \ddot{w}}{\partial x}\right) + \frac{1}{3}\frac{\partial \ddot{\phi}_z}{\partial x}\right] \tag{54}$$

$$\delta v: \quad \frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} + q_y = m_0 \ddot{v} + m_1 \ddot{\theta}_y - \frac{m_2}{2}\frac{\partial \ddot{\theta}_z}{\partial y} - m_3\left[c_1\left(\ddot{\theta}_y + \frac{\partial \ddot{w}}{\partial y}\right) + \frac{1}{3}\frac{\partial \ddot{\phi}_z}{\partial y}\right] \tag{55}$$

$$\begin{aligned}\delta w: \quad & \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}M_{xx}^{(0)} + \frac{\partial w}{\partial y}M_{xy}^{(0)}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}M_{xy}^{(0)} + \frac{\partial w}{\partial y}M_{yy}^{(0)}\right) \\ & + \frac{\partial M_{xz}^{(0)}}{\partial x} + \frac{\partial M_{yz}^{(0)}}{\partial y} + c_1\left(\frac{\partial^2 M_{xx}^{(3)}}{\partial x^2} + 2\frac{\partial^2 M_{xy}^{(3)}}{\partial x \partial y} + \frac{\partial^2 M_{yy}^{(3)}}{\partial y^2}\right) \\ & - c_2\left(\frac{\partial M_{xz}^{(2)}}{\partial x} + \frac{\partial M_{yz}^{(2)}}{\partial y}\right) + q_z \\ & = m_0 \ddot{w} + m_1 \ddot{\theta}_z + m_2 \ddot{\phi}_z \\ & + c_1\left[m_3\left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}\right) - c_1 m_6\left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2}\right) + \hat{m}_4\left(\frac{\partial \ddot{\theta}_x}{\partial x} + \frac{\partial \ddot{\theta}_y}{\partial y}\right)\right. \\ & \left. - \frac{m_5}{2}\left(\frac{\partial^2 \ddot{\theta}_z}{\partial x^2} + \frac{\partial^2 \ddot{\theta}_z}{\partial y^2}\right) - \frac{m_6}{3}\left(\frac{\partial^2 \ddot{\phi}_z}{\partial x^2} + \frac{\partial^2 \ddot{\phi}_z}{\partial y^2}\right)\right]\end{aligned} \tag{56}$$

$$\begin{aligned}\delta \theta_x: \quad & \frac{\partial M_{xx}^{(1)}}{\partial x} + \frac{\partial M_{xy}^{(1)}}{\partial y} - M_{xz}^{(0)} - c_1\left(\frac{\partial M_{xx}^{(3)}}{\partial x} + \frac{\partial M_{xy}^{(3)}}{\partial y}\right) + c_2 M_{xz}^{(2)} + F_x \\ & = \hat{m}_1 \ddot{u} + \hat{m}_2 \ddot{\theta}_x - \frac{1}{2}\hat{m}_3 \frac{\partial \ddot{\theta}_z}{\partial x} - \frac{1}{3}\hat{m}_4 \frac{\partial \ddot{\phi}_z}{\partial x} - c_1 \hat{m}_4\left(\ddot{\theta}_x + \frac{\partial \ddot{w}}{\partial x}\right)\end{aligned} \tag{57}$$

$$\begin{aligned}\delta \theta_y: \quad & \frac{\partial M_{xy}^{(1)}}{\partial x} + \frac{\partial M_{yy}^{(1)}}{\partial y} - M_{yz}^{(0)} - c_1\left(\frac{\partial M_{xy}^{(3)}}{\partial x} + \frac{\partial M_{yy}^{(3)}}{\partial y}\right) + c_2 M_{yz}^{(2)} + F_y \\ & = \hat{m}_1 \ddot{v} + \hat{m}_2 \ddot{\theta}_y - \frac{1}{2}\hat{m}_3 \frac{\partial \ddot{\theta}_z}{\partial y} - \frac{1}{3}\hat{m}_4 \frac{\partial \ddot{\phi}_z}{\partial y} - c_1 \hat{m}_4\left(\ddot{\theta}_y + \frac{\partial \ddot{w}}{\partial y}\right)\end{aligned} \tag{58}$$

$$\begin{aligned}\delta \theta_z: \quad & -M_{zz}^{(0)} + \frac{1}{2}\left(\frac{\partial^2 M_{xx}^{(2)}}{\partial x^2} + 2\frac{\partial^2 M_{xy}^{(2)}}{\partial x \partial y} + \frac{\partial^2 M_{yy}^{(2)}}{\partial y^2}\right) + P \\ & = \frac{1}{2}\left[m_2\left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}\right) + \hat{m}_3\left(\frac{\partial \ddot{\theta}_x}{\partial x} + \frac{\partial \ddot{\theta}_y}{\partial y}\right) - \frac{m_4}{2}\left(\frac{\partial^2 \ddot{\theta}_z}{\partial x^2} + \frac{\partial^2 \ddot{\theta}_z}{\partial y^2}\right)\right] \\ & + m_1 \ddot{w} + m_2 \ddot{\theta}_z + m_3 \ddot{\phi}_z\end{aligned} \tag{59}$$

$$\delta \phi_z: \quad -2M_{zz}^{(1)} + \frac{1}{3}\left(\frac{\partial^2 M_{xx}^{(3)}}{\partial x^2} + 2\frac{\partial^2 M_{xy}^{(3)}}{\partial x \partial y} + \frac{\partial^2 M_{yy}^{(3)}}{\partial y^2}\right) + Q$$

$$= m_2\ddot{w} + m_3\ddot{\theta}_z + m_4\ddot{\phi}_z + \frac{1}{3}\Bigg[m_3\left(\frac{\partial\ddot{u}}{\partial x} + \frac{\partial\ddot{v}}{\partial y}\right) - c_1 m_6\left(\frac{\partial^2\ddot{w}}{\partial x^2} + \frac{\partial^2\ddot{w}}{\partial y^2}\right)$$
$$+ \hat{m}_4\left(\frac{\partial\ddot{\theta}_x}{\partial x} + \frac{\partial\ddot{\theta}_y}{\partial y}\right) - \frac{m_5}{2}\left(\frac{\partial^2\ddot{\theta}_z}{\partial x^2} + \frac{\partial^2\ddot{\theta}_z}{\partial y^2}\right) - \frac{m_6}{3}\left(\frac{\partial^2\ddot{\phi}_z}{\partial x^2} + \frac{\partial^2\ddot{\phi}_z}{\partial y^2}\right)\Bigg] \tag{60}$$

where

$$\hat{m}_i = m_i - c_1 m_{i+2}, \quad i = 1, 2, 3, 4 \tag{61}$$

The boundary conditions involve specifying the following generalized forces that are dual to the generalized displacement $(u, v, w, \theta_x, \theta_y, \theta_z, \phi_z)$:

$$\delta u: \quad \bar{M}_{nn}^{(0)} \equiv M_{xx}^{(0)} n_x + M_{xy}^{(0)} n_y \tag{62}$$

$$\delta v: \quad \bar{M}_{nt}^{(0)} \equiv M_{xy}^{(0)} n_x + M_{yy}^{(0)} n_y \tag{63}$$

$$\delta w: \quad \bar{M}_{nz}^{(0)} \equiv \left(\frac{\partial w}{\partial x} M_{xx}^{(0)} + \frac{\partial w}{\partial y} M_{xy}^{(0)}\right) n_x + \left(\frac{\partial w}{\partial x} M_{xy}^{(0)} + \frac{\partial w}{\partial y} M_{yy}^{(0)}\right) n_y$$
$$+ M_{xz}^{(0)} n_x + M_{yz}^{(0)} n_y - c_2 (M_{xz}^{(2)} n_x + M_{yz}^{(2)} n_y)$$
$$+ c_1 \left[\left(\frac{\partial M_{xx}^{(3)}}{\partial x} + \frac{\partial M_{xy}^{(3)}}{\partial y}\right) n_x + \left(\frac{\partial M_{xy}^{(3)}}{\partial x} + \frac{\partial M_{yy}^{(3)}}{\partial y}\right) n_y\right]$$
$$+ c_1 \Bigg[\left(m_3\ddot{u} + \hat{m}_4\ddot{\theta}_x - \frac{m_5}{2}\frac{\partial\ddot{\theta}_z}{\partial x} - \frac{m_6}{3}\frac{\partial\ddot{\phi}_z}{\partial x} - c_1 m_6 \frac{\partial\ddot{w}}{\partial x}\right) n_x$$
$$+ \left(m_3\ddot{v} + \hat{m}_4\ddot{\theta}_y - \frac{m_5}{2}\frac{\partial\ddot{\theta}_z}{\partial y} - \frac{m_6}{3}\frac{\partial\ddot{\phi}_z}{\partial y} - c_1 m_6 \frac{\partial\ddot{w}}{\partial y}\right) n_y\Bigg] \tag{64}$$

$$\delta\theta_x: \quad \bar{M}_{nn}^{(1)} \equiv M_{xx}^{(1)} n_x + M_{xy}^{(1)} n_y - c_1 (M_{xx}^{(3)} n_x + M_{xy}^{(3)} n_y) \tag{65}$$

$$\delta\theta_y: \quad \bar{M}_{nt}^{(1)} \equiv M_{xy}^{(1)} n_x + M_{yy}^{(1)} n_y - c_1 (M_{xy}^{(3)} n_x + M_{yy}^{(3)} n_y) \tag{66}$$

$$\delta\theta_z: \quad \bar{M}_{nz}^{(1)} \equiv \left(\frac{\partial M_{xx}^{(2)}}{\partial x} + \frac{\partial M_{xy}^{(2)}}{\partial y}\right) n_x + \left(\frac{\partial M_{yy}^{(2)}}{\partial y} + \frac{\partial M_{xy}^{(2)}}{\partial x}\right) n_y$$
$$+ \left[m_2\ddot{u} + m_3\ddot{\theta}_x - \frac{m_4}{2}\frac{\partial\ddot{\theta}_z}{\partial x} - m_5\left(\frac{1}{3}\frac{\partial\ddot{\phi}_z}{\partial x} - c_1\ddot{\theta}_x - c_1\frac{\partial\ddot{w}}{\partial x}\right)\right] n_x$$
$$+ \left[m_2\ddot{v} + m_3\ddot{\theta}_y - \frac{m_4}{2}\frac{\partial\ddot{\theta}_z}{\partial y} - m_5\left(\frac{1}{3}\frac{\partial\ddot{\phi}_z}{\partial y} - c_1\ddot{\theta}_y - c_1\frac{\partial\ddot{w}}{\partial y}\right)\right] n_y \tag{67}$$

$$\delta\phi_z: \quad \bar{M}_{nz}^{(2)} \equiv \frac{2}{3}\left(\frac{\partial M_{xx}^{(3)}}{\partial x} + \frac{\partial M_{xy}^{(3)}}{\partial y}\right) n_x + \frac{2}{3}\left(\frac{\partial M_{yy}^{(3)}}{\partial y} + \frac{\partial M_{xy}^{(3)}}{\partial x}\right) n_y$$

$$+c_1\left[\left(m_3\ddot{u}+\hat{m}_4\ddot{\theta}_x-\frac{m_5}{2}\frac{\partial\ddot{\theta}_z}{\partial x}-\frac{m_6}{3}\frac{\partial\ddot{\phi}_z}{\partial x}-c_1m_6\frac{\partial\ddot{w}}{\partial x}\right)n_x\right.$$
$$\left.+\left(m_3\ddot{v}+\hat{m}_4\ddot{\theta}_y-\frac{m_5}{2}\frac{\partial\ddot{\theta}_z}{\partial y}-\frac{m_6}{3}\frac{\partial\ddot{\phi}_z}{\partial y}-c_1m_6\frac{\partial\ddot{w}}{\partial x}\right)n_y\right] \tag{68}$$

This third-order theory is not to be found in the literature; the closest one is that due to Reddy [1990].

4.2. *The Reddy third-order theory*

The Reddy third-order theory (Reddy [1984a]) is based on the displacement field in which $\theta_z = 0$ and $\phi_z = 0$; when the top and bottom surfaces of the plate are required to be free of any tangential forces, we obtain

$$\phi_x = 0, \quad \psi_x = -c_1\left(\theta_x + \frac{\partial w}{\partial x}\right), \quad \phi_y = 0, \quad \psi_y = -c_1\left(\theta_y + \frac{\partial w}{\partial y}\right) \tag{69}$$

Thus, the theory is a special case of the one derived in the previous section and it deduced by setting $\theta_z = 0$, $\phi_z = 0$, $\phi_x = 0$, and $\phi_y = 0$. Then the strains in Eq. (8)–(12) take the form

$$\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{Bmatrix} = \begin{Bmatrix}\varepsilon_{xx}^{(0)}\\ \varepsilon_{yy}^{(0)}\\ \gamma_{xy}^{(0)}\end{Bmatrix} + z\begin{Bmatrix}\varepsilon_{xx}^{(1)}\\ \varepsilon_{yy}^{(1)}\\ \gamma_{xy}^{(1)}\end{Bmatrix} + z^3\begin{Bmatrix}\varepsilon_{xx}^{(3)}\\ \varepsilon_{yy}^{(3)}\\ \gamma_{xy}^{(3)}\end{Bmatrix}, \quad \begin{Bmatrix}\varepsilon_{zz}\\ \gamma_{xz}\\ \gamma_{yz}\end{Bmatrix} = \begin{Bmatrix}\varepsilon_{zz}^{(0)}\\ \gamma_{xz}^{(0)}\\ \gamma_{yz}^{(0)}\end{Bmatrix} + z^2\begin{Bmatrix}\varepsilon_{zz}^{(2)}\\ \gamma_{xz}^{(2)}\\ \gamma_{yz}^{(2)}\end{Bmatrix} \tag{70}$$

with

$$\begin{Bmatrix}\varepsilon_{xx}^{(0)}\\ \varepsilon_{yy}^{(0)}\\ \gamma_{xy}^{(0)}\end{Bmatrix} = \begin{Bmatrix}\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\end{Bmatrix}, \quad \begin{Bmatrix}\varepsilon_{xx}^{(3)}\\ \varepsilon_{yy}^{(3)}\\ \gamma_{xy}^{(3)}\end{Bmatrix} = -c_1\begin{Bmatrix}\frac{\partial\theta_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\\ \frac{\partial\theta_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\\ \frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x} + 2\frac{\partial^2 w}{\partial x\partial y}\end{Bmatrix} \tag{71}$$

$$\begin{Bmatrix}\varepsilon_{xx}^{(1)}\\ \varepsilon_{yy}^{(1)}\\ \gamma_{xy}^{(1)}\end{Bmatrix} = \begin{Bmatrix}\frac{\partial\theta_x}{\partial x}\\ \frac{\partial\theta_y}{\partial y}\\ \frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\end{Bmatrix}, \quad \begin{Bmatrix}\varepsilon_{zz}^{(0)}\\ \gamma_{xz}^{(0)}\\ \gamma_{yz}^{(0)}\end{Bmatrix} = \begin{Bmatrix}0\\ \theta_x + \frac{\partial w}{\partial x}\\ \theta_y + \frac{\partial w}{\partial y}\end{Bmatrix}, \quad \begin{Bmatrix}\varepsilon_{zz}^{(2)}\\ \gamma_{xz}^{(2)}\\ \gamma_{yz}^{(2)}\end{Bmatrix} = -c_2\begin{Bmatrix}0\\ \theta_x + \frac{\partial w}{\partial x}\\ \theta_y + \frac{\partial w}{\partial y}\end{Bmatrix} \tag{72}$$

Thus the transverse normal strain ε_{zz} is identically zero. Consequently, σ_{zz} does not enter the strain energy expression and we use the plane stress-reduced constitutive relations

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & \nu & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \alpha\Delta T \\ \varepsilon_{yy} - \alpha\Delta T \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{73}$$

Thus, the stress resultants must be expressed in terms of the strains using the constitutive equations in Eq. (73).

The equations of motion of the Reddy third-order theory are

$$\delta u: \quad \frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} + q_x = m_0 \ddot{u} + m_1 \ddot{\theta}_x - m_3 c_1 \left(\ddot{\theta}_x + \frac{\partial \ddot{w}}{\partial x} \right) \tag{74}$$

$$\delta v: \quad \frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} + q_y = m_0 \ddot{v} + m_1 \ddot{\theta}_y - m_3 c_1 \left(\ddot{\theta}_y + \frac{\partial \ddot{w}}{\partial y} \right) \tag{75}$$

$$\begin{aligned} \delta w: \quad & \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} M_{xx}^{(0)} + \frac{\partial w}{\partial y} M_{xy}^{(0)} \right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x} M_{xy}^{(0)} + \frac{\partial w}{\partial y} M_{yy}^{(0)} \right) + \frac{\partial M_{xz}^{(0)}}{\partial x} + \frac{\partial M_{yz}^{(0)}}{\partial y} \\ & \quad + c_1 \left(\frac{\partial^2 M_{xx}^{(3)}}{\partial x^2} + 2\frac{\partial^2 M_{xy}^{(3)}}{\partial x \partial y} + \frac{\partial^2 M_{yy}^{(3)}}{\partial y^2} \right) - c_2 \left(\frac{\partial M_{xz}^{(2)}}{\partial x} + \frac{\partial M_{yz}^{(2)}}{\partial y} \right) + q_z \\ & = c_1 \left[m_3 \left(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y} \right) - c_1 m_6 \left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2} \right) + \hat{m}_4 \left(\frac{\partial \ddot{\theta}_x}{\partial x} + \frac{\partial \ddot{\theta}_y}{\partial y} \right) \right] \\ & \quad + m_0 \ddot{w} \end{aligned} \tag{76}$$

$$\begin{aligned} \delta \theta_x: \quad & \frac{\partial M_{xx}^{(1)}}{\partial x} + \frac{\partial M_{xy}^{(1)}}{\partial y} - M_{xz}^{(0)} - c_1 \left(\frac{\partial M_{xx}^{(3)}}{\partial x} + \frac{\partial M_{xy}^{(3)}}{\partial y} \right) + c_2 M_{xz}^{(2)} + F_x \\ & = \hat{m}_1 \ddot{u} + \hat{m}_2 \ddot{\theta}_x - c_1 \hat{m}_4 \left(\ddot{\theta}_x + \frac{\partial \ddot{w}}{\partial x} \right) \end{aligned} \tag{77}$$

$$\begin{aligned} \delta \theta_y: \quad & \frac{\partial M_{xy}^{(1)}}{\partial x} + \frac{\partial M_{yy}^{(1)}}{\partial y} - M_{yz}^{(0)} - c_1 \left(\frac{\partial M_{xy}^{(3)}}{\partial x} + \frac{\partial M_{yy}^{(3)}}{\partial y} \right) + c_2 M_{yz}^{(2)} + F_y \\ & = \hat{m}_1 \ddot{v} + \hat{m}_2 \ddot{\theta}_y - c_1 \hat{m}_4 \left(\ddot{\theta}_y + \frac{\partial \ddot{w}}{\partial y} \right) \end{aligned} \tag{78}$$

The generalized forces ($M_{xx}^{(i)}$, $M_{yy}^{(i)}$, $M_{xy}^{(i)}$) are related to the generalized displacements $(u, v, w, \theta_x, \theta_y)$ by Eqs. (48) with

$$\begin{aligned} A_{11}^{(k)} &= \frac{1}{(1-\nu^2)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z,T)\, dz, \quad A_{12}^{(k)} = \frac{\nu}{(1-\nu^2)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z,T)\, dz \\ B_{11}^{(k)} &= \frac{1}{2(1+\nu)} \int_{-\frac{h}{2}}^{\frac{h}{2}} (z)^k E(z,T)\, dz \end{aligned} \tag{79}$$

4.3. *The first-order plate theory*

The well-known first-order plate theory[a] is based on the displacement field in which we have $\theta_z = 0$, $\phi_z = 0$, $\phi_x = 0$, and $\phi_y = 0$ as well as $c_1 = 0$ and $c_2 = 0$. Then the strains in Eq. (70)–(72) simplify to

$$\left\{\begin{matrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{matrix}\right\} = \left\{\begin{matrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{matrix}\right\} + z \left\{\begin{matrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{matrix}\right\}, \quad \left\{\begin{matrix} \varepsilon_{zz} \\ \gamma_{xz} \\ \gamma_{yz} \end{matrix}\right\} = \left\{\begin{matrix} \varepsilon_{zz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ \theta_x + \frac{\partial w}{\partial x} \\ \theta_y + \frac{\partial w}{\partial y} \end{matrix}\right\} \tag{80}$$

with

$$\left\{\begin{matrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{matrix}\right\} = \left\{\begin{matrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{matrix}\right\}, \quad \left\{\begin{matrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{matrix}\right\} = \left\{\begin{matrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \end{matrix}\right\} \tag{81}$$

The equations of motion are simplified to

$$\delta u: \quad \frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} + q_x = m_0 \ddot{u} + m_1 \ddot{\theta}_x \tag{82}$$

$$\delta v: \quad \frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} + q_y = m_0 \ddot{v} + m_1 \ddot{\theta}_y \tag{83}$$

$$\begin{aligned} \delta w: \quad & \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} M_{xx}^{(0)} + \frac{\partial w}{\partial y} M_{xy}^{(0)}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x} M_{xy}^{(0)} + \frac{\partial w}{\partial y} M_{yy}^{(0)}\right) \\ & + \frac{\partial M_{xz}^{(0)}}{\partial x} + \frac{\partial M_{yz}^{(0)}}{\partial y} + q_z = m_0 \ddot{w} \end{aligned} \tag{84}$$

$$\delta \theta_x: \quad \frac{\partial M_{xx}^{(1)}}{\partial x} + \frac{\partial M_{xy}^{(1)}}{\partial y} - M_{xz}^{(0)} + F_x = m_1 \ddot{u} + m_2 \ddot{\theta}_x \tag{85}$$

$$\delta \theta_y: \quad \frac{\partial M_{xy}^{(1)}}{\partial x} + \frac{\partial M_{yy}^{(1)}}{\partial y} - M_{yz}^{(0)} + F_y = m_1 \ddot{v} + m_2 \ddot{\theta}_y \tag{86}$$

The generalized forces ($M_{xx}^{(i)}$, $M_{yy}^{(i)}$, $M_{xy}^{(i)}$) are related to the generalized displacements $(u, v, w, \theta_x, \theta_y)$ by Eqs. (48) with the coefficients $A_{11}^{(k)}$, $A_{12}^{(k)}$, and $B_{11}^{(k)}$ defined by Eq. (79).

4.4. *The classical plate theory*

The classical plate theory is obtained by setting

$$\theta_x = -\frac{\partial w}{\partial x}, \quad \theta_y = -\frac{\partial w}{\partial y} \tag{87}$$

[a] The author does not like to use the phrases *Mindlin plate theory* or *Reissner-Mindlin plate theory* and prefers to use the phrase *first-order plate theory* because Mindlin and Reissner were not the ones who developed the theory; it goes back to Cauchy, Basset, Hencky, and others (see Reddy [2004] for a review of the literature).

and $\phi_x = \phi_y = \psi_x = \psi_y = \theta_z = \phi_z = 0$ in the displacement field (6). The equations of motion are obtained from the Reddy third-order theory by setting $c_1 = 1$ and deleting terms involving $M_{xz}^{(0)}$ and $M_{yz}^{(0)}$. We have

$$\delta u: \quad \frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} + q_x = m_0 \ddot{u} - m_1 \frac{\partial \ddot{w}}{\partial x} \tag{88}$$

$$\delta v: \quad \frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} + q_y = m_0 \ddot{v} - m_1 \frac{\partial \ddot{w}}{\partial y} \tag{89}$$

$$\begin{aligned} \delta w: \quad & \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} M_{xx}^{(0)} + \frac{\partial w}{\partial y} M_{xy}^{(0)}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x} M_{xy}^{(0)} + \frac{\partial w}{\partial y} M_{yy}^{(0)}\right) + q_z \\ & + \frac{\partial^2 M_{xx}^{(1)}}{\partial x^2} + 2\frac{\partial^2 M_{xy}^{(1)}}{\partial x \partial y} + \frac{\partial^2 M_{yy}^{(1)}}{\partial y^2} = m_0 \ddot{w} + m_1 \left(\ddot{u} n_x + \ddot{v} n_y\right) \\ & + m_2 \left(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}_y}{\partial y^2}\right) \end{aligned} \tag{90}$$

5. Closing Comments

A general third-order theory of functionally graded plates is developed. The theory accounts for temperature dependent properties and the von Kármán nonlinearity. The equations of motions and associated force boundary conditions are derived using the principle of virtual displacements for the dynamic case. The theory contains 11 generalized displacements. Then three-dimensional constitutive relations for functionally graded plates are developed consistent with the third-order theory. The existing plate theories are deduced as special cases of the developed general third-order plate theory.

The general third-order theory developed herein can be used to develop finite element models of functionally graded plates. For the general case, the finite element models allow C^0-approximation of all 11 generalized displacements. The third-order plate theories with traction-free top and bottom surfaces require C^0 interpolation of $(u, v, \theta_x, \theta_y)$ and Hermite interpolation of w, θ_z, and ϕ_z. Computational models and their applications of the new general third-order theories presented here are yet to appear.

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